Persistent Random Walks. I. Recurrence Versus Transience

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Abstract We consider a walker on the line that at each step keeps the same direction with a probability which depends on the time already spent in the direction the walker is currently moving. These walks with memories of variable length can be seen as generalizations of Directionally Reinforced Random Walks (DRRWs) introduced in [1, Mauldin & al., Adv. Math., 1996]. We give a complete and usable characterization of the recurrence or transience in terms of the probabilities to switch the direction and we formulate some law of large numbers. The most fruitful situation emerges when the running times have both an infinite mean. In that case, these properties are related to the behaviour of some embedded random walk with an undefined drift so that these features depend on the asymptotics of the distribution tails related to the persistence times. In the other case, the criterion reduces to a null-drift condition. Finally, we deduce some criteria for a wider class of Persistent Random Walks (PRWs) whose increments are encoded by a Variable Length Markov Chain (VLMC) having – in full generality – no renewal pattern in such way that their study do not reduce to a skeleton RW as for the original model.

Key words Persistent and directionally reinforced random walks . Variable length memory . Recurrence and transience . Random walk with undefined mean

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Introduction

Classical random walks are usually defined from a sequence of independent and identically distributed (*i.i.d.*) increments $\{X_k\}_{k\geqslant 1}$ by

$$S_0 := 0 \quad \text{and} \quad S_n := \sum_{k=1}^{n} X_k \quad \text{for all integers} \quad n \geqslant 1. \tag{1.1}$$

When the jumps are defined as a finite-order Markov chain, a short memory in the dynamics of the stochastic paths is introduced and the random walk $\{S_n\}_{n\geqslant 0}$ itself is no longer Markovian. Such a process is called in the literature a PRW, a Goldstein-Kac random walk or also a correlated random walk. Concerning the genesis of the theory, we allude to [2–7] as regards the discrete-time situation but also its connections with the continuous-time telegraph process and we refer to [8, 9] concerning recurrence and transience features.

In this paper, we aim at investigating the asymptotic behaviour of one-dimensional PRW for which the increments are driven by a VLMC (an infinite-order Markov chain) built from a probabilized context tree. This construction furnishes an extented model for the dependence of the increments of PRWs which can be easily adapted to various situations. The slight presentation of VLMCs below – that fits our model – comes from [10] and we refer to [11, pp. 117-134] and [12] for an overview.

Let $\mathscr{L} = \mathscr{A}^{-\mathbb{N}}$ be the set of left-infinite words on the alphabet $\mathscr{A} := \{\mathtt{d}, \mathtt{u}\}$ and consider a complete tree on this alphabet, *i.e.* such that each node has 0 or 2 children, whose leaves $\mathscr{C}$ are words (possibly infinite) on $\mathscr{A}$. To each leaf $c \in \mathscr{C}$, called a context, is attached a probability distribution q_c on $\mathscr{A}$. Endowed with this probabilistic structure, such a tree is named a probabilized context tree. The related VLMC $\{U_n\}_{n\geqslant 0}$ is defined as the Markov Chain on $\mathscr{L}$ whose transitions are given by

$$\mathbb{P}(U_{n+1} = U_n \ell | U_n) = q_{\overleftarrow{\mathrm{pref}}\,(U_n)}(\ell), \tag{1.2}$$

where $\overleftarrow{\mathrm{pref}}\,(w) \in \mathscr{C}$ is defined as the shortest prefix of $w = \cdots w_{-1} w_0$, read from the right to the left, appearing as a leaf of the context tree. The kth increment X_k of the corresponding PRW is given as the rightmost letter of U_k, with the one-to-one correspondence $\mathtt{d} = -1$ (for a descent) and $\mathtt{u} = 1$ (for a rise).

Different probabilized context trees lead to different probabilistic impacts on the asymptotic behaviour of the resulting PRWs. Besides, the characterization of the recurrent *versus* transient behaviour, the so called type problem, is difficult for a general probabilized context tree. Here, we state exhaustive and handy criteria – in terms of the distribution tails of the persistent times – together with some law of large numbers for PRWs defined from a double-infinite comb introduced in [13]. Roughly speaking, the leaves – coding for the memory – are the words on $\{\mathtt{d}, \mathtt{u}\} \simeq \{-1, 1\}$ of the form $\mathtt{d}^n\mathtt{u}$ and $\mathtt{u}^n\mathtt{d}$. Hence, the probability to invert the current direction depends only on its length. In addition, we derive sufficient conditions for the type of PRWs built from a larger class of context trees obtained as grafts of the original double-infinite comb model.

Closely related to our model, DRRWs are nearest neighbour random walks keeping their directions during random times τ, independently and identically drawn after every change of directions, themselves chosen independently and uniformly among the other ones. In dimension one, the double-infinite comb model can be seen as a generalisation since it allows running times $\tau^{\mathtt{u}}$ (up) and $\tau^{\mathtt{d}}$ (down) with distinct distributions. Due to their symmetry, the recurrence criterion of DRRWs in dimension one takes the simple form given in [1, Theorem 3.1., p. 244] and obviously we retrieve this particular result in our more general situation (see Proposition 3.1 and Theorem 3.1). It is stated in [1, Theorem 3.3. and Theorem 3.4., p. 245] that these random walks are recurrent in $\mathbb{Z}^2$ when the waiting time between changes of direction is square integrable, and transient in $\mathbb{Z}^3$ when it is only supposed to be integrable. In higher dimension, it is shown that it is always transient. In dimension three the corresponding result has been recently improved in [14, Theorem 2., p. 682] by removing the integrability condition. Also, the assertion in [1] that the DRRW is transient when its embedded random walk of successive locations of change into the first direction is transient has been partially invalided in [14, Theorem 4., p. 684]. Thus, even in the symmetric situation, the characterization of recurrence or transience is a difficult task. The case of anisotropic PRWs built from VLMCs in higher dimension is a work in progress and this paper is somehow a first step, as the study of their scaling limits which will be presented in a forthcoming paper.

In Section 2, we briefly recall the double-infinite comb model of PRW and we introduce the major quantities and notations required in the sequel. The renewal property stated in [13] implies for the PRW infinitely many U-turns, cutting its paths into independent pieces. As a result, one may define the walk more directly and forget the underlying VLMC structure since the even (and odd) breaking times together with the position of the walk at these instants form two coupled classical RWs. However, in the last section, we get some criteria for a broader class of PRWs without renewal assumption and therefore motivating the use of VLMCs to model the dependance of the past. The penultimate section 3 is devoted to the recurrence or transience of the double-infinite comb PRW. Its main result – Theorem 3.1 – can be viewed as the continuation of Erickson's theorem in [15, 1974]. Besides, some extensions of the Strong Law of the Large Number (SLLN) shown in [13] are also given.

Settings and assumptions

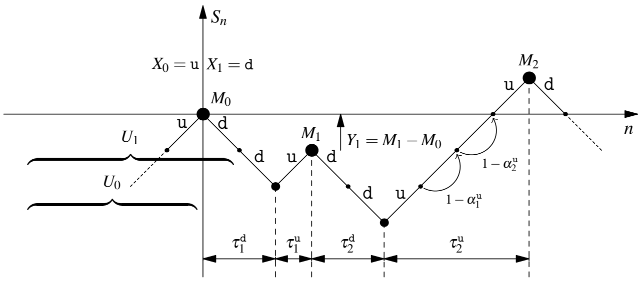

Figure 2.1: Persistent random walk

Foremost, we refer carefully to Figure 2.1 that illustrates our notations and assumptions by a realization of S, the so called double-infinite comb PRW. This PRW is characterized by the transition probabilities

$$\alpha_k^{\mathtt{d}} := \mathbb{P}(X_{k+1} = \mathtt{u} | X_0 = \mathtt{u}, X_1 = \mathtt{d}, \cdots, X_k = \mathtt{d})$$
$$\text{and} \quad \alpha_k^{\mathtt{u}} := \mathbb{P}(X_{k+1} = \mathtt{d} | X_0 = \mathtt{d}, X_1 = \mathtt{u}, \cdots, X_k = \mathtt{u}), \tag{2.1}$$

where X_k denotes the kth jump of S in $\{\mathtt{d}, \mathtt{u}\} \simeq \{-1, 1\}$ given as the rightmost letter of the left-infinite word U_k – the kth term of underlying double-infinite comb VLMC defined in [13]. Note that the latter conditional probabilities are invariant by shifting the sequence of increments and thus $\alpha_k^{\mathtt{u}}$ and $\alpha_k^{\mathtt{d}}$ stand respectively for the probabilities of changing direction after k rises and k descents.

Furthermore, in order to avoid trivial cases, we assume that S can not be frozen in one of the two directions with a positive probability – it makes infinitely many U-turns almost surely. Besides, without loss of generality, we deal throughout this paper with the conditional probability with respect to the event $(X_0, X_1) = (\mathtt{u}, \mathtt{d})$ – the initial time is suppose to be an up-to-down turn. Therefore, we assume the following.

Assumption 2.1 (finiteness of the length of runs)**.** *For any $\ell \in \{\mathtt{u}, \mathtt{d}\}$,*

$$\prod_{k=1}^{\infty} (1 - \alpha_k^{\ell}) = 0 \iff \left(\exists k \geqslant 1 \ \ s.t. \ \ \alpha_k^{\ell} = 1 \quad or \quad \sum_{k=1}^{\infty} \alpha_k^{\ell} = \infty \right). \tag{2.2}$$

Let $\tau_n^{\mathtt{u}}$ and $\tau_n^{\mathtt{d}}$ be respectively the length of the nth rise and of the nth descent (also called persistence times). Then, by the renewal property of the underlying VLMC, $\{\tau_n^{\mathtt{d}}\}$ and $\{\tau_n^{\mathtt{u}}\}$ are independent sequences of *i.i.d.* random variables. Their distribution tails are given for any $\ell \in \{\mathtt{u}, \mathtt{d}\}$ and $n \geqslant 1$ by

$$\mathscr{T}_\ell(n) := \mathbb{P}(\tau_1^{\ell} \geqslant n) = \prod_{k=1}^{n-1} (1 - \alpha_k^{\ell}). \tag{2.3}$$

At this stage, we exclude the situation of almost surely constant length of runs which trivializes the analysis. In order to deal with a more tractable random walk built with possibly unbounded but *i.i.d.* increments, we introduce the underlying skeleton random walk $\{M_n\}_{n \geqslant 0}$ associated with the even U-turns – the original walk observed at the random times of up-to-down turns. Note that the expectation $\mathbf{d}_M$ of an increment $Y_k := \tau_k^{\mathtt{u}} - \tau_k^{\mathtt{d}}$ is meaningful whenever one of the persistence times is integrable. In this situation, we can set (extended by continuity whenever necessary)

$$\mathbf{d}_S := \frac{\mathbb{E}[\tau_1^{\mathtt{u}}] - \mathbb{E}[\tau_1^{\mathtt{d}}]}{\mathbb{E}[\tau_1^{\mathtt{u}}] + \mathbb{E}[\tau_1^{\mathtt{d}}]} \in [-1, 1]. \tag{2.4}$$

In regards to the convergence (3.1) below, the latter quantity is naturally termed the almost sure drift of the double-infinite comb PRW.

Recurrence and transience

As stated by [16, Theorem 1., Chap. XII and Theorem 4., Chap. VI], any non-constant one-dimensional random walk is either, with probability one, oscillating – $\{-\infty,\infty\}$ are both limit points – or drifting toward $\pm\infty$. Moreover, whenever its increments satisfy some SLLN, these cases are characterized through the associated almost sure drift. In fact, the recurrence *versus* transience behaviour of the double-infinite comb PRW reduces to the oscillating *versus* drifting behaviour of its skeleton random walk.

Lemma 3.1 (equivalent characterization)**.** *The double-infinite comb PRW S is either recurrent or transient according to its skeleton M is oscillating or drifting.*

Proof. Obviously, if M is oscillating, then S is recurrent. Next, if M is drifting to $-\infty$, then S is transient to $-\infty$ since the trajectory of S is always under the broken line formed by the M_n's. Finally, noting that, up to an independent random variable, the skeleton random walk at odd breaking times (down-to-up breaking times) is equal in distribution to M, this ends the proof of the lemma. □

Lemma 3.2 (comparison lemma)**.** *Let S and $\widetilde{S}$ be two double-infinite comb PRWs such that the distribution tails of their runs satisfy $\mathscr{T}_{\mathrm{u}} \leqslant \widetilde{\mathscr{T}_{\mathrm{u}}}$ and $\mathscr{T}_{\mathrm{d}} \geqslant \widetilde{\mathscr{T}_{\mathrm{d}}}$. Then there exists a coupling – still denoted by $(S,\widetilde{S})$ – such that $S \leqslant \widetilde{S}$ a.s..*

Proof. Let G_ℓ and $\widetilde{G}_\ell$ be the left-continuous inverses of the cumulative distribution functions of the persistence times and introduce two independent sequences $\{V_n^\ell\}$ of uniform random variables on $[0,1]$. Since double-infinite comb PRWs are entirely determined by the lengths of runs, a coupling can be constructed by considering the sequences of running times given by $\tau_n^\ell := G_\ell(V_n^\ell)$ and $\widetilde{\tau}_n^\ell := \widetilde{G}_\ell(V_n^\ell)$. □

First assume $\mathbf{d}_S$ given in (2.4) is well-defined. In this case, a SLLN for the double-infinite comb PRW can be stated and the recurrence is characterized by a null-drift condition similarly to the classical context of random walks with integrable jumps.

Proposition 3.1 (well-defined drift case)**.** *The double-infinite comb PRW S is recurrent if and only if $\mathbf{d}_S = 0$ and transient otherwise. Furthermore, one has*

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbf{d}_S \quad a.s.. \tag{3.1}$$

Proof. Remark that the recurrence criterion is a straightforward consequence of [16, Theorem 1., Chap. XII and Theorem 4., Chap. VI] and Lemma 3.1 above. Besides, the law of large numbers (3.1) when $\mathbb{E}[\tau_1^{\mathrm{u}}]$ and $\mathbb{E}[\tau_1^{\mathrm{d}}]$ are both finite is already proved in [13, Proposition 4.5, p. 33]. Then (by symmetry) it only remains to prove the SLLN when $\mathbb{E}[\tau_1^{\mathrm{u}}] = \infty$ and $\mathbb{E}[\tau_1^{\mathrm{d}}] < \infty$ – the limit $\mathbf{d}_S$ being then equal to 1. Note that it is sufficient to prove the underestimate in (3.1). Let $N \geqslant 1$ and set

$$\alpha_n^{\mathrm{u},N} := \begin{cases} \alpha_n^{\mathrm{u}}, & \text{when} \quad 1 \leqslant n \leqslant N-1, \\ 1, & \text{when} \quad n \geqslant N, \end{cases} \qquad \text{and} \quad \alpha_n^{\mathrm{d},N} := \alpha_n^{\mathrm{d}}. \tag{3.2}$$

This defines a double-infinite comb PRW, denoted by $S^{[N]}$, satisfying $\mathbf{d}_{S^{[N]}} \geqslant 1-\varepsilon$ as soon as $N := N(\varepsilon)$ is chosen sufficiently large for any $\varepsilon \in (0,1)$. Then Lemma 3.2 gives a coupling such that $S^{[N]} \leqslant S$ a.s. and we deduce the result. □

We consider the remaining case in which both $\mathbb{E}[\tau_1^{\mathrm{u}}]$ and $\mathbb{E}[\tau_1^{\mathrm{d}}]$ are infinite. Following Erickson [15, Theorem 2., p. 372], the oscillating or drifting behaviour of the skeleton random walk M is characterized through the cumulative distribution function of its increments $Y_k = \tau_k^{\mathrm{u}} - \tau_k^{\mathrm{d}}$. However, Erickson's criterion does not suit to our context – without *ad hoc* regularity assumptions on the distributions – since the law of a jump is not explicitly given by the parameters of the model, but rather by the convolution of two *a priori* known distributions. More precisely, this criterion requires to settle whether the quantities

$$J_+ := \sum_{n=1}^{\infty} \frac{n\mathbb{P}(Y_1 = n)}{\sum_{k=1}^{n} \mathbb{P}(Y_1 \leqslant -k)} \quad \text{and} \quad J_- := \sum_{n=1}^{\infty} \frac{n\mathbb{P}(Y_1 = -n)}{\sum_{k=1}^{n} \mathbb{P}(Y_1 \geqslant k)}, \tag{3.3}$$

are finite or infinite, which is not convenient in concrete cases. To circumvent these difficulties, we consider a sequence $\{\xi_n\}$ of *i.i.d.* symmetric Bernoulli random variables, independent of the length of runs, and we set for any $n \geqslant 0$,

$$M_n^{[\xi]} := \sum_{k=1}^{n} Y_k^{[\xi]}, \quad \text{with} \quad Y_k^{[\xi]} := \xi_k \tau_k^{\mathtt{u}} - (1-\xi_k)\tau_k^{\mathtt{d}}. \tag{3.4}$$

The proof of the following Lemma is postponed to the end of this section.

Lemma 3.3 (randomized random walk)**.** *The random walks M and $M^{[\xi]}$ are simultaneously oscillating or drifting.*

Therefore, applying the criteria of Erickson to $M^{[\xi]}$, it is not difficult to see that it consists of studying the convergence of the more tractable series

$$J_{\ell_1|\ell_2} := \sum_{n=1}^{\infty} \frac{n\mathbb{P}(\tau^{\ell_1} = n)}{\sum_{k=1}^{n} \mathbb{P}(\tau^{\ell_2} \geqslant k)} = \sum_{n=1}^{\infty} \frac{n(-\Delta \mathscr{T}_{\ell_1}(n))}{\sum_{k=1}^{n} \mathscr{T}_{\ell_2}(k)}, \tag{3.5}$$

for any $\ell_1 \neq \ell_2 \in \{\mathtt{u},\mathtt{d}\}$, where $\Delta V(n) := V(n+1) - V(n)$. Again, the proof of our main result below is postponed to the end of this part.

Theorem 3.1 (undefined drift case)**.** *The double-infinite comb PRW S is recurrent if and only if $J_{\mathtt{u}|\mathtt{d}}$ and $J_{\mathtt{d}|\mathtt{u}}$ are both infinite. Otherwise, it is transient to ∞ (resp. $-\infty$) if and only if only $J_{\mathtt{u}|\mathtt{d}}$ (resp. $J_{\mathtt{d}|\mathtt{u}}$) is finite. In any case, when $J_{\mathtt{u}|\mathtt{d}} = \infty$ (resp. $J_{\mathtt{d}|\mathtt{u}} = \infty$),*

$$\limsup_{n\to\infty} \frac{S_n}{n} = 1 \quad a.s. \quad \left(resp. \;\; \liminf_{n\to\infty} \frac{S_n}{n} = -1 \quad a.s. \right). \tag{3.6}$$

The case of finite characteristics $J_{\mathtt{u}|\mathtt{d}}$ and $J_{\mathtt{d}|\mathtt{u}}$ does not appear in this theorem because then it follows from [15] that the persistence times are both integrable. Hence, this case reduces to the well-defined drift case in Proposition 3.1. Besides, this theorem can be reformulated in terms of alternative quantities involving only the distribution tails of the running times, making the criterion more transparent.

Corollary 3.1 (alternative formulation)**.** *Theorem 3.1 can be stated in terms of the characteristics $K_{\ell_1|\ell_2}$ in place of $J_{\ell_1|\ell_2}$ with*

$$K_{\ell_1|\ell_2} := \sum_{n=1}^{\infty} \left(1 - \frac{n\mathscr{T}_{\ell_2}(n)}{\sum_{k=1}^{n} \mathscr{T}_{\ell_2}(k)}\right) \frac{\mathscr{T}_{\ell_1}(n)}{\sum_{k=1}^{n} \mathscr{T}_{\ell_2}(k)}. \tag{3.7}$$

Proof. By symmetry, we only need to prove that $J_{\mathtt{u}|\mathtt{d}} = \infty$ if and only if $K_{\mathtt{u}|\mathtt{d}} = \infty$. Summing by parts – the so called Abel transformation – we can write for any $r \geqslant 1$,

$$\sum_{n=1}^{r} \frac{n(-\Delta \mathscr{T}_{\mathtt{u}}(n))}{\sum_{k=1}^{n} \mathscr{T}_{\mathtt{d}}(k)} = \left[1 - \frac{(r+1)\mathscr{T}_{\mathtt{u}}(r+1)}{\sum_{k=1}^{r+1} \mathscr{T}_{\mathtt{d}}(k)}\right] + \sum_{n=1}^{r} \Delta\left(\frac{n}{\sum_{k=1}^{n} \mathscr{T}_{\mathtt{d}}(k)}\right) \mathscr{T}_{\mathtt{u}}(n+1).$$

Besides, a simple computation gives

$$\Delta\left(\frac{n}{\sum_{k=1}^{n} \mathscr{T}_{\mathtt{d}}(k)}\right) = \frac{\sum_{k=1}^{n} \mathscr{T}_{\mathtt{d}}(k) - n\mathscr{T}_{\mathtt{d}}(n+1)}{\sum_{k=1}^{n+1} \mathscr{T}_{\mathtt{d}}(k) \sum_{k=1}^{n} \mathscr{T}_{\mathtt{d}}(k)} = \frac{\mathbb{E}[\tau^{\mathtt{d}} \mathbb{1}_{\tau_1^{\mathtt{d}} \leqslant n}]}{\sum_{k=1}^{n+1} \mathscr{T}_{\mathtt{d}}(k) \sum_{k=1}^{n} \mathscr{T}_{\mathtt{d}}(k)} \geqslant 0. \tag{3.8}$$

It follows that

$$\sum_{n=1}^{r}\frac{n(-\Delta\mathscr{T}_{\mathtt{u}}(n))}{\sum_{k=1}^{n}\mathscr{T}_{\mathtt{d}}(k)}=\left[1-\frac{(r+1)\mathscr{T}_{\mathtt{u}}(r+1)}{\sum_{k=1}^{r+1}\mathscr{T}_{\mathtt{d}}(k)}\right]+\sum_{n=1}^{r}\left(1-\frac{n\mathscr{T}_{\mathtt{d}}(n+1)}{\sum_{k=1}^{n}\mathscr{T}_{\mathtt{d}}(k)}\right)\frac{\mathscr{T}_{\mathtt{u}}(n+1)}{\sum_{k=1}^{n+1}\mathscr{T}_{\mathtt{d}}(k)}. \tag{3.9}$$

Due to (3.8) and the non-integrability of the persistence times, the general term of the series in the right-hand side of the latter equation is non-negative and (up to a shift) equivalent to that of $K_{\mathtt{u}|\mathtt{d}}$. As a consequence, if $J_{\mathtt{u}|\mathtt{d}}$ is infinite, then so is $K_{\mathtt{u}|\mathtt{d}}$. Moreover, we get again from (3.8) that

$$\frac{r\mathscr{T}_{\mathtt{u}}(r)}{\sum_{k=1}^{r}\mathscr{T}_{\mathtt{d}}(k)}=\sum_{m=r+1}^{\infty}\frac{r(-\Delta\mathscr{T}_{\mathtt{u}}(m))}{\sum_{k=1}^{r}\mathscr{T}_{\mathtt{d}}(k)}\leqslant\sum_{m=r+1}^{\infty}\frac{m(-\Delta\mathscr{T}_{\mathtt{u}}(m))}{\sum_{k=1}^{m}\mathscr{T}_{\mathtt{d}}(k)}. \tag{3.10}$$

Therefore, the finiteness of $J_{\mathtt{u}|\mathtt{d}}$ and (3.10) imply the first term on the right-hand side in (3.9) is bounded and thus the finiteness of $K_{\mathtt{u}|\mathtt{d}}$. This ends the proof of the corollary. □

Theorem 3.1. First, the statement related to the recurrence and the transience are direct consequences of Erickson's criteria [15] and of Lemma 3.3. Then, assume that $J_{\mathtt{u}|\mathtt{d}}=\infty$. Note that the left-hand side of (3.6) is satisfied if, for all $c>0$,

$$\mathbb{P}\left(\tau_n^{\mathtt{u}}\geqslant c\sum_{k=1}^{n}\tau_k^{\mathtt{d}}\quad i.o.\right)=1. \tag{3.11}$$

By using the Kolmogorov's zero-one law, we only need to prove that this probability is not zero. To this end, remark [17, Theorem 5., p. 1190] applies to $M^{[\xi]}$ so that

$$\limsup_{n\to\infty}\frac{(Y_n^{[\xi]})^+}{\sum_{k=1}^{n}(Y_n^{[\xi]})^-}=\limsup_{n\to\infty}\frac{\xi_n\tau_n^{\mathtt{u}}}{\sum_{k=1}^{n}(1-\xi_k)\tau_k^{\mathtt{d}}}=\infty\quad a.s.. \tag{3.12}$$

Roughly speaking, this theorem states that the position of a one-dimensional random walk with an undefined mean is essentially given by the last big jump. Introducing the counting process $N_n:=\mathrm{card}\{1\leqslant k\leqslant n:\xi_k=0\}$, we shall prove that

$$\left\{\sum_{k=1}^{n}(1-\xi_k)\tau_k^{\mathtt{d}}\right\}_{n\geqslant1}\overset{\mathscr{L}}{=}\left\{\sum_{k=1}^{N_n}\tau_k^{\mathtt{d}}\right\}_{n\geqslant1}. \tag{3.13}$$

We will check that $\{(1-\xi_n)\tau_n^{\mathtt{d}}\}$ and $\{(1-\xi_n)\tau_{N_n}^{\mathtt{d}}\}$ are sequences of independent random variables with marginals equal in distribution. First note that for any $n\geqslant1$,

$$\mathbb{P}((1-\xi_n)\tau_{N_n}^{\mathtt{d}}=0)=\mathbb{P}(\xi_1=1)=\mathbb{P}((1-\xi_n)\tau_n^{\mathtt{d}}=0). \tag{3.14}$$

Moreover, up to a null set, we have $\{\xi_n=0\}=\{N_n=N_{n-1}+1\}$ and N_{n-1} is independent of ξ_n and of the lengths of runs. We deduce that for any $k\geqslant1$,

$$\mathbb{P}((1-\xi_n)\tau_{N_n}^{\mathtt{d}}=k)=\mathbb{P}(\xi_1=0,\tau_1^{\mathtt{d}}=k)=\mathbb{P}((1-\xi_n)\tau_n^{\mathtt{d}}=k). \tag{3.15}$$

Hence, the increments of the random walks in (3.13) are identically distributed and it only remains to prove the independence. Fix $n\geqslant1$ and set for any $k_1,\cdots,k_n\geqslant0$, $I_n:=\{1\leqslant j\leqslant n:k_j\neq0\}$ and $m_n:=\mathrm{card}(I_n)$. Remark that $\ell\longmapsto m_\ell$ is increasing on I_n and up to a null set,

$$\bigcap_{\ell\notin I_n}\{\xi_\ell=1\}\cap\bigcap_{\ell\in I_n}\{\xi_\ell=0\}\subset\{N_n=m_n\}. \tag{3.16}$$

Then using (3.14) and (3.15) together with the independence properties we get

$$\mathbb{P}\left(\bigcap_{j=1}^{n}\{(1-\xi_j)\tau^{\mathtt{d}}_{N_j}=k_j\}\right)=\mathbb{P}\left(\bigcap_{\ell\notin I_n}\{\xi_\ell=1\}\cap\bigcap_{\ell\in I_n}\{\xi_\ell=0,\tau^{\mathtt{d}}_{m_\ell}=k_\ell\}\right)$$
$$=\prod_{j=1}^{n}\mathbb{P}((1-\xi_j)\tau^{\mathtt{d}}_{N_j}=k_j),$$

which ends the proof of (3.13). Furthermore, by the SLLN, we obtain that for any integer $q\geqslant 2$, with probability one, the events $\{N_n\geqslant\lfloor n/q\rfloor\}$ hold for all n sufficiently large. Therefore, we deduce from (3.13) and (3.12) that

$$\left\{\tau^{\mathtt{u}}_n\geqslant c\sum_{k=1}^{\lfloor n/q\rfloor}\tau^{\mathtt{d}}_k\quad i.o.\right\}\quad\text{and thus}\quad\left\{\bigcup_{\ell=0}^{q-1}\left\{\tau^{\mathtt{u}}_{qn+\ell}\geqslant c\sum_{k=1}^{n}\tau^{\mathtt{d}}_k\right\}\quad i.o.\right\},$$

are events of probability one. Again, applying the Kolmogorov's zero-one law, we get that the q sequences of events (having the same distribution) in the latter equation occur infinitely often. This achieves the proof of (3.11) and – by symmetry – of (3.6). This completes the proof of the theorem. □

Remark 3.1. *The reccurence and transience criteria in Theroren 3.1 relies on the probabilistic proof of Lemma 3.3 presented below. However, one can wonder whether a more analytic proof is feasible in order to replace the characteristics (3.3) deduced from [15] by those given in (3.5). Unfortunately, we have been unable to find such a proof except under the regularity assumption*

$$\sup\left\{\frac{\max\{\mathbb{P}(\tau^{\ell}_1=k):k\geqslant n\}}{\mathbb{P}(\tau^{\ell}_1=n)}:n\geqslant 1,\ell\in\{\mathtt{u},\mathtt{d}\}\right\}<\infty. \tag{3.17}$$

Lemma 3.3. We deeply exploit the structure of one-dimensional random walks stating that they are either oscillating or drifting to $\pm\infty$. Assume that the supremum limit of $M^{[\xi]}$ is *a.s.* infinite. Then, following exactly the same lines as in the proof of (3.11) we obtain that the supremum limit of M is also *a.s.* infinite. Thereafter, by symmetry and the structure theorem, we only need to prove that if $M^{[\xi]}$ is drifting to ∞, then so is M. Furthermore, since the *i.i.d.* Bernoulli random variables $\{\xi_n\}$ are symmetric – that is $p=1/2$ – it is a simple consequence of the equalities

$$M^{[\xi]}\stackrel{\mathscr{L}}{=}M^{[1-\xi]}\quad\text{and}\quad M=M^{[\xi]}+M^{[1-\xi]}. \tag{3.18}$$

Here $1-\xi:=\{1-\xi_n\}$ is the complementary sequence of *i.i.d.* symmetric Bernoulli random variables of $\xi=\{\xi_n\}$. □

Remark 3.2. *Lemma 3.3 still holds when $\{\xi_n\}$ is supposed to be an i.i.d. sequence of Bernoulli random variables of parameter $p\in(0,1)$. The proof is based on the case $p=1/2$ and on the arguments in the proof of (3.6).*

Remark 3.3. *Contrary to the well-defined drift case, here double-infinite comb PRWs may stay recurrent when the α^{ℓ}_k's are slightly perturbed. To put it in a nutshell, the criterion is global in the former case and asymptotic in the latter case.*

On arbitrary PRWs

Consider a double-infinite comb and attach to each finite leaf c another context tree $\mathbb{T}_c$ (possibly trivial) as in Figure 4.2. The leaves of the related graft are denoted by $\mathscr{C}_c$ and this one is endowed with Bernoulli distributions $\{q_l : l \in \mathscr{C}_c\}$ on $\{\mathtt{u},\mathtt{d}\}$. Note that any probabilized context tree on $\{\mathtt{u},\mathtt{d}\}$ can be constructed in this way. We denote by S^{g} the corresponding PRW. In this case, the random walk is particularly persistent in the sense that the rises and descents are no longer independent. A renewal property may still hold but it is more tedious to express in general. Let $\underline{S}$ and $\overline{S}$ be the double-infinite comb PRWs with respective transition

$$\underline{\alpha}_n^{\mathtt{u}} := \sup\{q_c(\mathtt{d}) : c \in \mathscr{C}_{\mathtt{u}^n\mathtt{d}}\}, \quad \underline{\alpha}_n^{\mathtt{d}} = \inf\{q_c(\mathtt{u}) : c \in \mathscr{C}_{\mathtt{d}^n\mathtt{u}}\},$$
$$\text{and} \quad \overline{\alpha}_n^{\mathtt{u}} := \inf\{q_c(\mathtt{d}) : c \in \mathscr{C}_{\mathtt{u}^n\mathtt{d}}\}, \quad \overline{\alpha}_n^{\mathtt{d}} := \sup\{q_c(\mathtt{u}) : c \in \mathscr{C}_{\mathtt{d}^n\mathtt{u}}\}. \tag{4.1}$$

Proposition 4.1. *Assume that $\underline{S}$ and $\overline{S}$ have persistent times of infinite mean in such a way that they are simultaneously recurrent or transient. Then, S^{g} is of the same type.*

For instance, when the (non-trivial) grafts are some finite trees in finite number such that the attached Bernoulli distributions are non-degenerated and induce non-integrable running times, recurrence or transience criteria reduce to the study of some double-infinite comb PRW.

This proposition is a straightforward consequence of Lemma 4.1 and Lemma 4.2 below which are of independent interest. Lemma 4.1 gives sufficient condition for the recurrence or the transience of an arbitrary PRW. Its proof involves a coupling argument similar to the one appearing in the proof of Lemma 3.2 and is therefore omitted. Lemma 4.2, on the other hand, implements the Remark 3.3 above.

Lemma 4.1 (comparison of grafts)**.** *With probability one, one has*

$$\limsup_{n\to\infty} \underline{S}_n = \infty \implies \limsup_{n\to\infty} S_n^{\mathrm{g}} = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \overline{S}_n = -\infty \implies \liminf_{n\to\infty} S_n^{\mathrm{g}} = -\infty.$$

As a consequence, if $\underline{S}$ and $\overline{S}$ are of the same recurrent or transient type, then S is recurrent or transient accordingly.

Lemma 4.2 (asymptotic comparison)**.** *Let S and $\widetilde{S}$ be two double-infinite comb PRWs with non-integrable running times. If their exists $c \geqslant 1$ such that their distribution tails satisfy, for n large enough,*

$$\mathscr{T}_{\mathtt{u}}(n) \leqslant c\widetilde{\mathscr{T}}_{\mathtt{u}}(n) \quad \textit{and} \quad \mathscr{T}_{\mathtt{d}}(n) \geqslant c^{-1}\widetilde{\mathscr{T}}_{\mathtt{d}}(n), \tag{4.2}$$

then

$$\limsup_{n\to\infty} S_n = \infty \quad \textit{a.s.} \quad \implies \quad \limsup_{n\to\infty} \widetilde{S}_n = \infty \quad \textit{a.s.}. \tag{4.3}$$

Proof. Let $N \geqslant 1$ be such that $c\widetilde{\mathscr{T}}_{\mathtt{u}}(N) \leqslant 1$ together with (4.2) for all $n \geqslant N$. Then

$$\widehat{\mathscr{T}}_{\mathtt{u}}(n) := \begin{cases} c\widetilde{\mathscr{T}}_{\mathtt{u}}(n), & \text{if} \quad n \geqslant N, \\ 1, & \text{if} \quad n < N, \end{cases} \quad \text{and} \quad \widehat{\mathscr{T}}_{\mathtt{d}}(n) := \begin{cases} c^{-1}\widetilde{\mathscr{T}}_{\mathtt{d}}(n), & \text{if} \quad n \geqslant N, \\ \mathscr{T}_{\mathtt{d}}(n), & \text{if} \quad n < N, \end{cases}$$

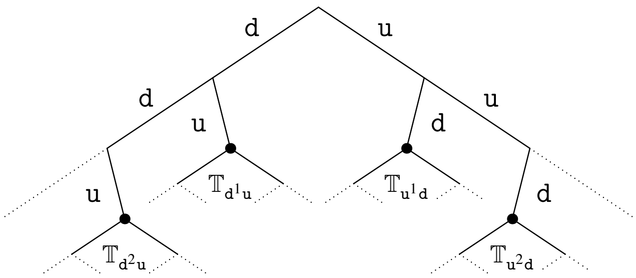

Figure 4.2: Grafting of the double-infinite comb

are distribution tails of some runs associated with a double-comb PRW $\widehat{S}$. Due to Lemma 3.2, there exists a coupling such that $S \leqslant \widehat{S}$ and it follows from Theorem 3.1 that (4.3) holds with $\widehat{S}$ in place of $\widetilde{S}$. Therefore, it suffices to show that the hat and tilde $K_{\mathtt{u}|\mathtt{d}}$-characteristics in (3.7) are simultaneously finite or infinite. First note that

$$\frac{\widehat{\mathscr{T}}_{\mathtt{u}}(n)}{\left(\sum_{k=1}^{n} \widehat{\mathscr{T}}_{\mathtt{d}}(k)\right)^2} \underset{n\to\infty}{\sim} c^3 \frac{\widetilde{\mathscr{T}}_{\mathtt{u}}(n)}{\left(\sum_{k=1}^{n} \widetilde{\mathscr{T}}_{\mathtt{d}}(k)\right)^2},$$

since the hat and the tilde distribution tails only differ for finitely many n and because the denominators in the latter equation tends to infinity by hypothesis. Besides,

$$\left(\sum_{k=1}^{n} \widehat{\mathscr{T}}_{\mathtt{d}}(k)\right) - n\widehat{\mathscr{T}}_{\mathtt{d}}(n) \underset{n\to\infty}{\sim} c^{-1}\left[\left(\sum_{k=1}^{n} \widetilde{\mathscr{T}}_{\mathtt{d}}(k)\right) - n\widetilde{\mathscr{T}}_{\mathtt{d}}(n)\right]$$

by noting that these quantities are nothing but the truncated means of the lengths of runs and thus go to infinity (always by assumption). Consequently, the proof follows from the two latter equations and the expression given in (3.7). □

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References

[1] R. Daniel Mauldin, Michael Monticino, and Heinrich von Weizsäcker. Directionally reinforced random walks. *Adv. Math.*, 117(2):239–252, 1996. 1, 2

[2] S. Goldstein. On diffusion by discontinuous movements, and on the telegraph equation. *Quart. J. Mech. Appl. Math.*, 4:129–156, 1951. 1

[3] Mark Kac. A stochastic model related to the telegrapher's equation. *Rocky Mountain J. Math.*, 4:497–509, 1974. Reprinting of an article published in 1956, Papers arising from a Conference on Stochastic Differential Equations (Univ. Alberta, Edmonton, Alta., 1972). 1

[4] Eric Renshaw and Robin Henderson. The correlated random walk. *J. Appl. Probab.*, 18(2):403–414, 1981. 1

[5] George H. Weiss. *Aspects and applications of the random walk*. Random Materials and Processes. North-Holland Publishing Co., Amsterdam, 1994. 1

[6] Eugene C. Eckstein, Jerome A. Goldstein, and Mark Leggas. The mathematics of suspensions: Kac walks and asymptotic analyticity. In *Proceedings of the Fourth Mississippi State Conference on Difference Equations and Computational Simulations (1999)*, volume 3 of *Electron. J. Differ. Equ. Conf.*, pages 39–50. Southwest Texas State Univ., San Marcos, TX, 2000. 1

[7] George H. Weiss. Some applications of persistent random walks and the telegrapher's equation. *Phys. A*, 311(3-4):381–410, 2002. 1

[8] Robin Henderson, Eric Renshaw, and David Ford. A note on the recurrence of a correlated random walk. *J. Appl. Probab.*, 20(3):696–699, 1983. 1

[9] Marco Lenci. Recurrence for persistent random walks in two dimensions. *Stoch. Dyn.*, 7(1):53–74, 2007. 1

[10] Peggy Cénac, Brigitte Chauvin, Frédéric Paccaut, and Nicolas Pouyanne. Context trees, variable length Markov chains and dynamical sources. In *Séminaire de Probabilités XLIV*, volume 2046 of *Lecture Notes in Math.*, pages 1–39. Springer, Heidelberg, 2012. 2

[11] Imre Csiszár. Festschrift in honor of Jorma Rissanen on the occasion of his 75th birthday edited by Peter Grünwald, Petri Myllymäki, Ioan Tabus, Marcelo Weinberger, Bin Yu. *International Statistical Review*, 76(3):436–437, 2008. 2

[12] Jorma Rissanen. A universal data compression system. *IEEE Trans. Inform. Theory*, 29(5):656–664, 1983. 2

[13] P. Cénac, B. Chauvin, S. Herrmann, and P. Vallois. Persistent random walks, variable length Markov chains and piecewise deterministic Markov processes. *Markov Process. Related Fields*, 19(1):1–50, 2013. 2, 3, 4

[14] Rainer Siegmund-Schultze and Heinrich von Weizsäcker. Level crossing probabilities. II. Polygonal recurrence of multidimensional random walks. *Adv. Math.*, 208(2):680–698, 2007. 2

[15] K. Bruce Erickson. The strong law of large numbers when the mean is undefined. *Trans. Amer. Math. Soc.*, 185:371–381 (1974), 1973. 2, 4, 5, 6, 7

[16] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971. 4

[17] Harry Kesten. The limit points of a normalized random walk. *Ann. Math. Statist.*, 41:1173–1205, 1970. 6