Persistent random walks. II. Functional Scaling Limits

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Abstract In this paper, we aim at giving a complete and unified description of the functional scaling limits associated with some Persistent Random Walks (PRW) for which the recurrence features are studied in [1]. As a result, we highlight a phase transition phenomenon with respect to the memory: depending on the decreasing rate of the tail distribution of the persistence times, the resulting limit process is Markovian or non-Markovian. A relevant generalized drift, appearing as a mean drift in full generality, is introduced to unify [the stability assumptions and] these two regimes. In the memoryless situation, the limits are classical strictly stable Lévy processes of infinite variations. Let us point out that the description of the critical Cauchy case, as well as the non-symmetric situation, fills some lacuna of the litterature. More specifically, for the closely related context of Directionally Reinforced Random Walks (DRRWs) in [2, 3], those cases have not been considered. Furthermore, the limit processes keeping some memory has been introduced in [4] as scaling limits of some Lévy Walks. We extend their results to our model but also to a wider class of PRWs without renewal patterns. Besides, we compute explicitly the marginal densities. To this end, we make the connection with the occupation times of some stochastic processes modelled on those defined in [5] and [included ???] skew Bessel processes. Finally, we clarify some misunderstanding regarding the latter marginal distributions in the framework of DRRWs and LWs.

Key words persistent random walks . functional scaling limits . arcsine Lamperti distributions . directionally reinforced random walk . Lévy walk . anomalous diffusion

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1 Introduction

This paper is a continuation of [1], in which recurrence versus transience features of some Persistent Random Walks (PRWs) are described. More specifically, we still consider a walker $\{S_n\}_{n\geqslant 0}$ on $\mathbb{Z}$, whose jumps are of unit size, and such that at each step, it keeps the same direction (or switches) with a probability directly depending on the time already spent in the direction the walker is currently moving. Here, we aim at investigating functional scaling limits of the form

$$\left\{\frac{S_{\lfloor ut\rfloor}-\mathbf{m}_s\, ut}{\lambda(u)}\right\}_{t\geqslant 0} \quad \text{or} \quad \left\{\frac{S_{ut}-\mathbf{m}_s\, ut}{\lambda(u)}\right\}_{t\geqslant 0} \xrightarrow[u\to\infty]{\mathscr{L}} \{Z(t)\}_{t\geqslant 0}, \tag{1.1}$$

for which some functional convergence in distribution toward a stochastic process Z holds. The continuous time stochastic process $\{S_t\}_{t\geqslant 0}$ above denotes the piecewise linear interpolation of the discrete time one $\{S_n\}_{n\geqslant 0}$. Due to the sizes of its jumps, the [latter???] is obviously ballistic or sub-ballistic. In particular, the drift parameter $\mathbf{m}_s$ belongs to $[-1,1]$ and the growth rate of the normalizing positive function $\lambda(u)$ is at most linear. In full generality, we aim at investigating a PRW given by

$$S_0 = 0 \quad \text{and} \quad S_n := \sum_{k=1}^{n} X_k, \quad \text{for all } n \geqslant 1, \tag{1.2}$$

where a two-sided process of jumps $\{X_k\}_{n\in\mathbb{Z}}$ in an additive group G is considered. In order to take into account possibly infinite reinforcements, the increment process is supposed to have a finite but possibly unbounded variable memory. More precisely, we assume that it is built from a Variable Length Markov Chain (VLMC) induced by some probabilized context tree. This construction furnishes extended models for the dependence of the increments which can be easily adapted to various situations.

Regarding the toy model roughly introduced above, the set of leaves $\mathscr{C}$ of the associated context tree, coding the memory, is nothing but the set of words $\{\mathtt{d}^n\mathtt{u},\, \mathtt{u}^n\mathtt{d} : n \geqslant 1\}$ with the correspondence $\mathtt{d} = -1$ (for a descent) and $\mathtt{u} = 1$ (for a rise). The associated two-sided process of jumps has a finite but possibly unbounded variable memory whose successive lengths are given by the so called age time process defined for all $n \geqslant 0$ by

$$A_n := \inf\{k \geqslant 1 : X_n \cdots X_{n-k} \in \mathscr{C}\}. \tag{1.3}$$

In particular, the PRW is completely characterized by the transition probabilities

$$\alpha_k^{\mathtt{d}} := \mathbb{P}(X_{n+1} = \mathtt{u} | X_n \cdots X_{n-k} = \mathtt{d}^k\mathtt{u}) = \mathbb{P}\left(X_{n+1} = \mathtt{u} | X_n = \mathtt{d}, A_n = k\right), \tag{1.4}$$

and

$$\alpha_k^{\mathtt{u}} := \mathbb{P}(X_{n+1} = \mathtt{d} | X_n \cdots X_{n-k} = \mathtt{u}^k\mathtt{d}) = \mathbb{P}\left(X_{n+1} = \mathtt{d} | X_n = \mathtt{u}, A_n = k\right). \tag{1.5}$$

It is worth noting that the dynamics can equivalently be described with the help of the distributions of the length associated with a typical rise and a typical descent. Those random variables are denoted by $\tau^{\mathtt{u}}$ and $\tau^{\mathtt{d}}$ respectively and represented in Figure 1.1. The related distribution tails and truncated means are given for every $\ell \in \{\mathtt{u},\mathtt{d}\}$ and $t \geqslant 0$ by

$$\mathscr{T}_\ell(t) := \mathbb{P}(\tau^\ell > t) = \prod_{k=1}^{\lfloor t\rfloor}(1-\alpha_k^\ell) \quad \text{and} \quad \Theta_\ell(t) := \mathbb{E}[\tau^\ell \wedge t] = \sum_{n=1}^{\lfloor t\rfloor}\prod_{k=1}^{n-1}(1-\alpha_k^\ell). \tag{1.6}$$

Implicitly, we exclude two pathological but irrelevant cases for which both of the length of runs are almost surely constant or infinite with positive probability. Besides, we can assume $(X_0, X_1) = (\mathtt{u}, \mathtt{d})$ without lose of generality. In the sequel, we also need to consider the truncated second moments

$$V_\ell(t) := \mathbb{E}[(\tau^\ell)^2 \mathbb{1}_{\{|\tau^\ell| \leqslant t\}}]. \tag{1.7}$$

Remark 1.1. *It is shown in [6] that the underlying VLMC admits a unique stationary probability measure if and only if the Markovian process $\{(X_n, A_n)\}_{n\geqslant 0}$ on $\{\mathtt{u}, \mathtt{d}\} \times \mathbb{Z}^+$ does. Besides, it is equivalent $\mathbb{E}[\tau^{\mathtt{d}}] < \infty$ and $\mathbb{E}[\tau^{\mathtt{u}}] < \infty$.*

1.1 Outline of the article

The paper is organized as follows: Section 1.2 is devoted to the definition of the mean drift $\mathbf{m}_S$ in (1.1) involved in the main Assumption 1.1. In Section 1.3 is defined the normalizing function $\lambda(u)$ appearing in Theorem 1.1 stating a generic version of the main results of this paper. More specific statements are given in Theorems 2.1 and 3.1 in Sections 2 and 3 respectively. The aim of Section 4 is two folds: first it is briefly shown that Theorem 3.1 can be extended to PRWs built from a wider class of probabilized context trees consisting of slight modifications of the original double infinite comb; secondly, the extremal case excluded by 1.1 is briefly discussed.

The proofs of Theorem 2.1 and 3.1 are given in Sections 2 and 3 respectively. Some standard estimates related to Assumption 1.1 are implicitly used in these proofs. For the sake of the reader, those estimates are recalled in A.1. Finally, in Appendix A.2, the notion of anomalous diffusion introduced in Section 3 is rigourously defined.

1.2 Mean drift and stability assumption

To begin with, when $\mathbb{E}[\tau^{\mathtt{d}}]$ or $\mathbb{E}[\tau^{\mathtt{u}}]$ is finite, we introduce

$$\mathbf{d}_S = \frac{\mathbf{d}_M}{\mathbf{d}_T} := \frac{\mathbb{E}[\tau^{\mathtt{u}}] - \mathbb{E}[\tau^{\mathtt{d}}]}{\mathbb{E}[\tau^{\mathtt{u}}] + \mathbb{E}[\tau^{\mathtt{d}}]}, \tag{1.8}$$

extended by continuity to ± 1 when only one of the persistence times has a finite expectation. This quantity naturally arises in the recurrence features of S as an almost sure drift as explained in [1]. Moreover, the quantities $\mathbf{d}_M$ and $\mathbf{d}_T$ are nothing but the usual drifts associated with the RWs considered in (2.6).

As another important quantity, there is the tail balance parameter defined as the following limit, when it makes sense,

$$\mathbf{b}_S := \lim_{t\to\infty} \frac{\mathscr{T}_{\mathtt{u}}(t) - \mathscr{T}_{\mathtt{d}}(t)}{\mathscr{T}_{\mathtt{u}}(t) + \mathscr{T}_{\mathtt{d}}(t)}. \tag{1.9}$$

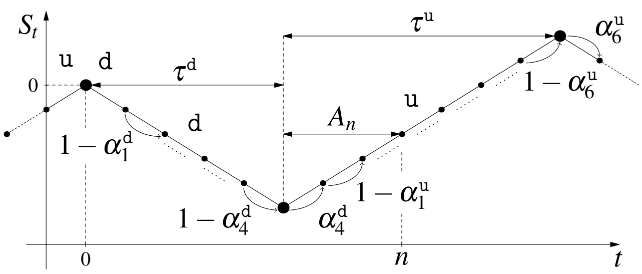

Figure 1.1: A sample path

Finally, let us set

$$\mathbf{m}_S := \begin{cases} \mathbf{b}_S, & \text{when } \tau^{\mathrm{u}} \text{ and } \tau^{\mathrm{d}} \text{ are both not integrable,} \\ \mathbf{d}_S, & \text{otherwise.} \end{cases} \tag{1.10}$$

In the light of the $\mathbb{L}^1$-convergence in (3.9), the latter quantity is naturally called, when it exists, the mean drift of S. This relevant characteristic (which is new to our knowledge) extends the notion of the almost-sure drift $\mathbf{d}_S$ since it can be rewritten as

$$\mathbf{m}_S = \lim_{t\to\infty} \frac{\Theta_{\mathrm{u}}(t) - \Theta_{\mathrm{d}}(t)}{\Theta_{\mathrm{u}}(t) + \Theta_{\mathrm{d}}(t)}. \tag{1.11}$$

Furthermore, a Strong Law of Large Number (SLLN) is established in [6], as well as a non-functional Central Limit Theorem (CLT), under strong moment conditions on τ^{d} and τ^{u}. Here, the assumptions are drastically weakened. More precisely, our main hypothesis is the following.

Assumption 1.1 (mean drift existence and α-stability)**.** *The mean drift* $\mathbf{m}_S$ *is well defined and not extremal, that is* $\mathbf{m}_S \in (-1,1)$. *Moreover, there exists* $\alpha \in (0,2]$ *such that*

$$\tau^{\mathrm{c}} := (1-\mathbf{m}_S)\tau^{\mathrm{u}} - (1+\mathbf{m}_S)\tau^{\mathrm{d}} \in \mathrm{D}(\alpha), \tag{1.12}$$

i.e. τ^{c} *belongs to the domain of attraction of an* α*-stable distribution.*

Obviously, the stable distribution of the assumption is supposed to be non-degenerated. For some additional characterizations of $\mathbf{m}_S$ and (1.12) we refer to Appendix A.1.

Remark 1.2. *If* τ^{u} *and* τ^{d} *are supposed to be identically distributed, the resulting PRWs belong to the family of the so-called DRRW processes introduced in [7] and motivated by the modelling of the ocean surface wave fields. Thus, PRWs considered here are nothing but the anisotropic extension of DRRWs and* $\mathbf{m}_S$ *can be interpreted as a quantification of this asymmetry.*

1.3 Statements of the results

We shall establish a general functional invariance principle, stated in its compact form in Theorem 1.1 below, and refined in Theorems 2.1 and 3.1. For this purpose, it is needed to introduce the suitable normalizing function $\lambda(u)$ in (1.1). To this end, consider the non-negative and non-decreasing functions $\Sigma(t)$ and $\Theta(t)$ given by

$$\Sigma(t)^2 := (1-\mathbf{m}_S)^2 V_{\mathrm{u}}\left(\frac{t}{1-\mathbf{m}_S}\right) + (1+\mathbf{m}_S)^2 V_{\mathrm{d}}\left(\frac{t}{1+\mathbf{m}_S}\right) \quad \text{and} \quad \Theta(t) := \Theta_{\mathrm{u}}(t) + \Theta_{\mathrm{d}}(t). \tag{1.13}$$

Thereafter, we set $\lambda(u) := a \circ s(u)$, with

$$a(u) := \inf\left\{t > 0 : \frac{t^2}{\Sigma(t)^2} \geqslant u\right\} \quad \text{and} \quad s(u) := \inf\{t > 0 : \Theta \circ a(t) t \geqslant u\}. \tag{1.14}$$

Again, we refer to Appendix A.1 [leading to???] the regular variations of these normalization functions. In particular, it follows from classical properties of these functions (see [8, 9] for instance) that

$$\frac{a(u)}{(\Sigma \circ a(u))^2} \underset{u\to\infty}{\sim} u \quad \text{and} \quad \Theta \circ a \circ s(u) s(u) \underset{u\to\infty}{\sim} u. \tag{1.15}$$

Then, in virtue of the Slutsky's theorem [10, Theorem 4.4., p. 27], one can replace the four functions above by any other equivalent ones in a neighbourhood of infinity and obtain exactly the same functional scaling limits as in Theorems 1.1. In this theorem, the Skorohod space of all right-continuous functions $\omega : [0,\infty) \longrightarrow \mathbb{R}$ having left-limits (*càdlàg*) is endowed with the M_1-topology and the convergence in distribution with respect to the related Borel σ-fields is denoted by M_1 (see [11] for more details).

Theorem 1.1. *Under Assumption 1.1, there exists a non-trivial càdlàg stochastic process Z_α such that*

$$\left\{\frac{S_{\lfloor ut\rfloor}-\mathbf{m}_s ut}{\lambda(u)}\right\}_{t\geqslant 0} \quad \textit{and} \quad \left\{\frac{S_{ut}-\mathbf{m}_s ut}{\lambda(u)}\right\}_{t\geqslant 0} \xrightarrow[u\to\infty]{\mathrm{M}_1} \{Z_\alpha(t)\}_{t\geqslant 0}. \tag{1.16}$$

As a matter of fact, there are mainly four situations depending on the position of $\alpha \in (0,2]$ with respect to the partition $(0,1)\sqcup\{1\}\sqcup(1,2)\sqcup\{2\}$. More precisely, from right to left, the limit stochastic process Z_α is equal to

1) B – a standard Brownian motion;

2) $S_{\alpha,\beta}$ – a strictly α-stable Lévy process with skewness parameter

$$\beta = \frac{(1-\mathbf{d}_s)^\alpha(1+\mathbf{b}_s)-(1+\mathbf{d}_s)^\alpha(1-\mathbf{b}_s)}{(1-\mathbf{d}_s)^\alpha(1+\mathbf{b}_s)+(1+\mathbf{d}_s)^\alpha(1-\mathbf{b}_s)}, \tag{1.17}$$

   and Lévy jump measure

$$\left[\left(\frac{1-\beta}{2}\right)\mathbb{1}_{\{x<0\}}+\left(\frac{1+\beta}{2}\right)\mathbb{1}_{\{x>0\}}\right]\frac{(2-\alpha)}{|x|^{\alpha+1}}\,dx; \tag{1.18}$$

3) $\mathfrak{C}$ – a symmetric Cauchy process having the marginal distribution

$$\mathfrak{C}(1)\sim\frac{1}{2}\frac{dx}{(\pi/2)^2+x^2}; \tag{1.19}$$

4) $\mathscr{S}_\alpha$ – the arcsine Lamperti anomalous diffusion defined in Appendix A.2 (actually up to an affine transformation).

The first three situations cover all classical strictly stable Lévy processes having infinite variations. In Section 3 precisions are given regarding the notion of anomalous diffusion and the arcsine Lamperti denomination. This last case reprensents undoubtedly the most fruitful situation.

To put it in a nutshell, we bring out a phase transition phenomenon concerning the memory of the limit process. It is either a Lévy motion, so that the long term memory of the underlying PRW is lost after the change of scale, or it keeps an unbounded memory. In that case, this stochastic process is self-similar of index 1 with compactly supported marginal distributions and somehow ballistic since $\mathbb{V}(\mathscr{S}_\alpha(t)) = Dt^2$ for some positive constant D depending on α and $\mathbf{m}_s$. In full generality, anomalous diffusions (relatively to their mean squared displacement) often appear in the field of Continuous Time Random Walks (CTRWs) named also renewal reward processes or Lévy walks according to the context. Roughly speaking, those are nothing but classical RWs subordinated to a counting process. Their scaling limits has been widely investigated in [12–16]. Following [4] and particularly the "true" Lévy walks denomination, one can seen a DRRW, or more generally a PRW, as a linear interpolation of a CTRW.

Remark 1.3. *The authors in [6] show the convergence of some rescaling toward a generalized telegraph process. Here, the transitions (1.4) and (1.5) are fixed, genuine scaling limits are investigated.*

Remark 1.4. *The cases of $\alpha = 1$ or $\mathbf{m}_s \neq 0$ are not investigated in [2, 3]. Furthermore, it seems there are some misunderstandings in their results and proofs when $\alpha \in (0,1)$. For more details, we refer to Remark A.2 of the Appendix.*

Remark 1.5. *We make the effort to give an unified functional convergence. To enforce the scaling limit, we do not have to know* a priori *the index of stability α since computing the mean drift $\mathbf{m}_s$ and the normalization function $\lambda(u)$ only involve the truncated means and second moments of the persistence times. This could have some statistical interests.*

Theorem 1.1 is divided, completed and proved below according to these two regimes. Since each of the functional convergences in (1.16) implies the other one, Theorem 1.1 is a direct consequence of Theorems 2.1 and 3.1 together with Lemmas 2.1 and 3.1.

2 The Lévy situations

The following lemma gives precise asymptotic of the the normalizing functions.

Lemma 2.1. *For every $\alpha \in [1,2]$, there exists positive slowly varying functions $\Xi_\alpha(u)$ and $d(u)$, the latter being non-decreasing and converging toward $\mathbf{d}_T$, such that*

$$a(u) \underset{u\to\infty}{\sim} \Xi_\alpha(u)\, u^{1/\alpha} \quad \text{and} \quad s(u) \underset{u\to\infty}{\sim} \frac{u}{d(u)}, \quad \text{with} \quad \lim_{u\to\infty} \frac{d(u)^{1/\alpha}\, u^{1-1/\alpha}}{\Xi_\alpha(u/d(u))} = \infty \tag{2.1}$$

If $\alpha = 2$, one can choose for $\Xi_2(u)$ the non-decreasing function $\Sigma \circ a(u)$.

In the sequel, we consider stronger convergence in distribution: the J_1-convergence and its restriction $\mathscr{C}$ to the Wiener space of all continuous functions (see [10] for instance).

Theorem 2.1 (Lévy situations)**.** *For every $\alpha \in [1,2]$, the scaling limit (1.16) can be rewritten in a stronger way as follows.*

1) If $\alpha = 2$ then

$$\left\{ \frac{\sqrt{\mathbf{d}_T\, u}}{\Xi_2(u)} \left(\frac{S_{\lfloor ut \rfloor}}{u} - \mathbf{d}_S\, t \right) \right\}_{t\geqslant 0} \xRightarrow[u\to\infty]{\mathrm{J}_1} \{B(t)\}_{t\geqslant 0},$$
$$\text{and} \quad \left\{ \frac{\sqrt{\mathbf{d}_T\, u}}{\Xi_2(u)} \left(\frac{S_{ut}}{u} - \mathbf{d}_S\, t \right) \right\}_{t\geqslant 0} \xRightarrow[u\to\infty]{\mathscr{C}} \{B(t)\}_{t\geqslant 0}. \tag{2.2}$$

2) If $\alpha \in (1,2)$ then

$$\left\{ \frac{\mathbf{d}_T^{1/\alpha}\, u^{1-1/\alpha}}{\Xi_\alpha(u)} \left(\frac{S_{\lfloor ut \rfloor}}{u} - \mathbf{d}_S\, t \right) \right\}_{t\geqslant 0} \xRightarrow[u\to\infty]{\mathrm{J}_1} \{S_{\alpha,\beta}(t)\}_{t\geqslant 0}. \tag{2.3}$$

3) If $\alpha = 1$ then

$$\left\{ \frac{\mathbf{d}_T}{\Xi_1(u)} \left(\frac{S_{\lfloor ut \rfloor}}{u} - \mathbf{d}_S\, t \right) \right\}_{t\geqslant 0} \xRightarrow[u\to\infty]{\mathrm{J}_1} \{\mathfrak{C}(t)\}_{t\geqslant 0},$$
$$\text{or} \quad \left\{ \frac{d(u)}{\Xi_1(u/d(u))} \left(\frac{S_{\lfloor ut \rfloor}}{u} - \mathbf{b}_S\, t \right) \right\}_{t\geqslant 0} \xRightarrow[u\to\infty]{\mathrm{J}_1} \{\mathfrak{C}(t)\}_{t\geqslant 0}, \tag{2.4}$$

according to $\mathbf{d}_T < \infty$ or $\mathbf{d}_T = \infty$.

Remark 2.1. *Obviously, when the persistence times are both square integrable, one can replace $\Xi_2(u)$ in (2.2) by the the standard deviation of τ^c to retrieve the standard CLT obtained in [6].*

In this context, the quantity $\mathbf{m}_S$ appears as a drift in probability since

$$\lim_{n\to\infty} \frac{S_n}{n} \overset{\mathbb{P}}{=} \mathbf{m}_S. \tag{2.5}$$

Indeed, the latter convergence is a straightforward consequence of the SLLN proved in [1] when the waiting times are both integrable. On the contrary, it is completely new when $\mathbf{m}_S = \mathbf{b}_S$ and it simply follows from the right hand side of (2.4) since by Lemma 2.1 the multiplicative term tends to infinity.

Below, we prove Theorem 2.1. As a first step, Lemma 2.1 is admitted and its proof is postponed to the end of this section. Besides, we chiefly insist on the most difficult situation, the Cauchy one, never exposed to our knowledge.

Proofs

Let us introduce $\{\tau_k^{\mathrm{u}}\}_{k\geqslant 1}$ – the successive length of rises – and $\{\tau_k^{\mathrm{d}}\}_{k\geqslant 1}$ – the successive length of descents – associated with the PRW. Those are two independent sequences of *i.i.d.* random variables distributed as τ^{u} and τ^{d} respectively. Thereafter, we consider the sequence $\{T_n\}_{n\geqslant 0}$ of up-to-down breaking times and $\{M_n\}_{n\geqslant 0}$ the PRW observed at these moving times, both given by

$$T_n = \sum_{k=1}^{n}(\tau_k^{\mathrm{d}} + \tau_k^{\mathrm{u}}) \quad \text{and} \quad M_n = S_{T_n} = \sum_{k=1}^{n}(\tau_k^{\mathrm{d}} - \tau_k^{\mathrm{u}}), \tag{2.6}$$

with $T_0 = M_0 = 0$. Finally, we consider the continuous time counting process $\{N_t\}_{t\geqslant 0}$ associated with the non-decreasing random walk $\{T_n\}_{n\geqslant 0}$, that is

$$N(t) := \max\{n \geqslant 0 : T_n \leqslant t\} = \inf\{n \geqslant 0 : T_{n+1} > t\}, \tag{2.7}$$

The main idea consists of taking advantage of the following two decompositions, denoted respectively by **A** or **B**, accordingly the situations. We recall – see Appendix A.1 – that $\mathbf{m}_S = \mathbf{d}_S$ when $\alpha \in (1,2]$ or when $\alpha = 1$ and $\mathbf{d}_T < \infty$, whereas $\mathbf{m}_S = \mathbf{b}_S$ when $\alpha = 1$ and $\mathbf{d}_T = \infty$. In particular, the random walks in (2.8) and (2.11) below are *a priori* distinct.

A. When $\alpha \in (1,2]$ or when $\alpha = 1$ and $\mathbf{d}_T = \infty$, we consider the random walk $\{C_n\}_{n\geqslant 0}$ defined by $C_0 = 0$ and for every $n \geqslant 1$,

$$C_n := \sum_{k=1}^{n}\left[(1-\mathbf{m}_S)\tau_k^{\mathrm{u}} - (1+\mathbf{m}_S)\tau_k^{\mathrm{d}}\right]. \tag{2.8}$$

Thereafter, one can write the identity

$$S_{\lfloor ut\rfloor} - \mathbf{m}_S ut = \left(M_{N(ut)} - \mathbf{m}_S T_{N(ut)}\right) + R(ut) = C_{N(ut)} + R(ut), \tag{2.9}$$

where $\{R(v)\}_{v\geqslant 0}$ is a residual continuous time process satisfying

$$|R(v)| \leqslant (1-\mathbf{m}_S)\tau_{N(v)+1}^{\mathrm{u}} + (1+\mathbf{m}_S)\tau_{N(v)+1}^{\mathrm{d}}. \tag{2.10}$$

B. On the contrary, when $\alpha = 1$ and $\mathbf{d}_T < \infty$, we consider the random walk $\{\widetilde{C}_n\}_{n\geqslant 0}$ defined by $C_0 = 0$ and for every $n \geqslant 1$,

$$\widetilde{C}_n := \sum_{k=1}^{n}\left[(1-\mathbf{b}_S)\tau_k^{\mathrm{u}} - (1+\mathbf{b}_S)\tau_k^{\mathrm{d}}\right]. \tag{2.11}$$

Here, one can write the identity

$$S_{\lfloor ut\rfloor} - \mathbf{m}_S ut = \left(S_{\lfloor ut\rfloor} - \mathbf{b}_S ut\right) - (\mathbf{d}_S - \mathbf{b}_S)ut = \widetilde{C}_{N(ut)} - (\mathbf{d}_S - \mathbf{b}_S)ut + \widetilde{R}(ut). \tag{2.12}$$

Once again, the residual continuous time process $\{\widetilde{R}(v)\}_{v\geqslant 0}$ satisfies an upper bound similar to (2.10) but with $\mathbf{b}_S$ in place of $\mathbf{m}_S$. Those are obtained by straightforward computations.

The proof is organized as follows.

Step 1. We observe the following functional convergences.

A. According to $\alpha = 2$, $\alpha \in (1,2)$ or $\alpha = 1$ and $\mathbf{d}_T = \infty$,

$$\left\{\frac{C_{\lfloor ut\rfloor}}{a(u)}\right\}_{t\geqslant 0} \xrightarrow[u\to\infty]{\mathrm{J}_1} \{B(t)\}_{t\geqslant 0}, \quad \{S_{\alpha,\beta}(t)\}_{t\geqslant 0} \quad \text{or} \quad \{\mathfrak{C}(t)\}_{t\geqslant 0}. \tag{2.13}$$

B. On the contrary, if $\alpha = 1$ and $\mathbf{d}_T < \infty$, then

$$\left\{ \frac{\widetilde{C}_{\lfloor ut \rfloor} - (\mathbf{d}_S - \mathbf{b}_S)\, \mathbf{d}_T\, ut}{a(u)} \right\}_{t \geqslant 0} \xrightarrow[u\to\infty]{\mathrm{J}_1} \{\mathfrak{C}(t)\}_{t\geqslant 0}. \tag{2.14}$$

Step 2. In any case, for all $v \geqslant 0$,

$$\sup_{0\leqslant t\leqslant v} \left| \frac{N(ut)}{s(u)} - t \right| \xrightarrow[u\to\infty]{\mathbb{P}} 0. \tag{2.15}$$

Step 3. According to the cases **A** or **B**, we show that for all $v \geqslant 0$,

$$\sup_{0\leqslant t\leqslant v} \left| \frac{R(ut)}{a\circ s(u)} \right| \xrightarrow[u\to\infty]{\mathbb{P}} 0 \quad \text{or} \quad \sup_{0\leqslant t\leqslant v} \left| \frac{\widetilde{R}(ut)}{a\circ s(u)} \right| \xrightarrow[u\to\infty]{\mathbb{P}} 0. \tag{2.16}$$

Step 4. We conclude applying a classical continuous mapping theorem and the Slutsky's one.

We supply below the proofs of the four latter steps. Again, we fully used the standard estimations and remarks in Appendix A.1.

Step 1. In the case of $\alpha \in (1,2)$, the functional convergence in (2.13) is a direct consequence of Assumption 1.1 together with classical results which can be found in [17] or [11, Chap. 4.] for instance. To be more precise, we first need to note that τ^{c} is centered, since $\mathbf{d}_T < \infty$ and thus $\mathbf{m}_S = \mathbf{d}_S$, but also that the skewness parameter β in (1.17) is nothing but

$$\lim_{t\to\infty} \frac{\mathbb{P}(\tau^{\mathrm{c}} > t) - \mathbb{P}(\tau^{\mathrm{c}} < -t)}{\mathbb{P}(\tau^{\mathrm{c}} > t) + \mathbb{P}(\tau^{\mathrm{c}} < -t)}. \tag{2.17}$$

The latter equality is a consequence of (A.7). Then, it remains to check that $a(u)$ is the suitable normalizing function leading to the properly scaled α-stable Lévy process characterized by the Lévy jump measure (1.18). This follows from (A.8) and classical results on stable distributions in [8, 9]. As far as $\alpha = 2$, similar arguments holds, the skewness parameter being irrelevant. However, when $\alpha = 1$, we need the following Lemma whose proof is postponed to the end of this section.

Lemma 2.2. *Let W be a random walk whose jump w belongs to the domain of attraction of a symmetric Cauchy distribution. Then, the centering term in the scaling limit can be chosen to be equal to zero when w is not integrable, or to the drift otherwise, in such way that*

$$\left\{ \frac{W_{\lfloor ut \rfloor}}{k(u)} \right\}_{t\geqslant 0} \quad \text{or} \quad \left\{ \frac{W_{\lfloor ut \rfloor} - \mathbb{E}[w]\, ut}{k(u)} \right\}_{n\geqslant 0} \xrightarrow[u\to\infty]{\mathrm{J}_1} \{\mathfrak{C}(t)\}_{t\geqslant 0}, \tag{2.18}$$

accordingly. Here we denote by $k(u)$ the suitable normalizing function so that the limit process is the symmetric Cauchy process defined previously.

Therefore, when $\alpha = 1$ and $\mathbf{d}_T = \infty$, one can see that τ^{c} belongs to the domain of attraction of a symmetric Cauchy distribution since $\mathbf{m}_S = \mathbf{b}_S$ and

$$\lim_{t\to\infty} \frac{\mathbb{P}(\tau^{\mathrm{c}} > t) - \mathbb{P}(\tau^{\mathrm{c}} < -t)}{\mathbb{P}(\tau^{\mathrm{c}} > t) + \mathbb{P}(\tau^{\mathrm{c}} < -t)} = 0. \tag{2.19}$$

Again, the latter comes from (A.7). Then, applying the left hand side of (2.18) the proof follows the same lines as previously. Unfortunately, τ^{c} is no longer well balanced when $\alpha = 1$ and $\mathbf{d}_T < \infty$ because it is possible that $\mathbf{m}_S = \mathbf{d}_S \neq \mathbf{b}_S$. This is the reason why we consider $\widetilde{\tau}^{\mathrm{c}} := (1-\mathbf{b}_S)\tau^{\mathrm{u}} - (1-\mathbf{b}_S)\tau^{\mathrm{d}}$ which is still well balanced. However, we lose the the centering condition since

$$\mathbb{E}[\widetilde{\tau}^{\mathrm{c}}] = (\mathbf{d}_S - \mathbf{b}_S)\mathbf{d}_T. \tag{2.20}$$

Nevertheless, the right-hand side of (2.18) allow us to deduce the expected scaling limit (2.14).

Step 2. Again, we need to adapt the arguments accordingly as the persistence times are integrable or not. Using the switching identity $\{T_n \leqslant t\} = \{n \leqslant N(t)\}$, it suffices to show

$$\sup_{0\leqslant t\leqslant v} \left| \frac{T_{\lfloor s(u)t \rfloor}}{u} - t \right| \xrightarrow[u\to\infty]{\mathbb{P}} 0. \tag{2.21}$$

To this end, we first note that, if $\mathbf{d}_T < \infty$, the convergence in (2.21) at any fixed time t but without the supremum immediately follows from the weak law of large numbers, since from Lemma 2.1 one can chose $s(u) = u/\mathbf{d}_T$. On the other hand, the same marginal convergences still hold when $\mathbf{d}_T = \infty$ (and thus necessarily $\alpha = 1$) because in any case $\{T_n\}_{n\geqslant 0}$ is relatively stable with the inverse of $u \mapsto s(u)$ as normalizing function. Indeed, applying [18, p. 174] for instance, we obtain the convergence in distribution as u tends to infinity of

$$\frac{T_{\lfloor ut \rfloor}}{a(u)} - \frac{\Theta \circ a(u)\, u}{a(u)} t = \frac{\Theta \circ a(u)\, u}{a(u)} \left(\frac{T_{\lfloor ut \rfloor}}{\Theta \circ a(u)\, u} - t \right), \tag{2.22}$$

Since the involved functions are regularly varying, we get from Lemma 2.1 that

$$\Theta \circ a(s(u))\, s(u) \underset{u\to\infty}{\sim} u \quad \text{and} \quad a(u) \underset{u\to\infty}{\ll} \Theta \circ a(u)\, u. \tag{2.23}$$

This implies the convergence of the marginal distributions in (2.21) so that it only remains to prove the convergence is uniform.

It is well-known for stable distributions that the convergence of one marginal distribution implies its functional counterpart. Here the same is true. One way to prove this is to use the characteristics of semi-martingales which are in addition processes with independent increments. More precisely, applying the results in [19, Chap. VII, pp. 409-414], we only need to prove the uniform convergence on compact sets of the characteristics $(b_u, \tilde{c}_u, \nu_u)$ of $\{T_{\lfloor s(u)t \rfloor}/u\}_{t\geqslant 0}$ as u tends to infinity. Besides, following [19, Chap. II, pp. 94-96], one has for any truncation function h,

$$b_u(t) = \mathbb{E}\left[h\left(\frac{\tau_1}{u}\right)\right]\lfloor ut \rfloor, \quad \tilde{c}_u(t) = \mathbb{V}\left[h\left(\frac{\tau_1}{u}\right)\right]\lfloor ut \rfloor \quad \text{and} \quad \nu_u([0,t]\times g) := \mathbb{E}\left[g\left(\frac{\tau_1}{u}\right)\right]\lfloor ut \rfloor, \tag{2.24}$$

where $\tau_1 = \tau_1^{\mathrm{d}} + \tau_1^{\mathrm{u}}$ is a jump of $\{T_n\}_{n\geqslant 0}$ and g is any bounded continuous bounded by $x \mapsto x^2$ in a neighbourhood of the origin. The punctual convergence of this triplet toward $(t \mapsto t, 0, 0)$ follows from the convergence for $t = 1$. As a consequence, its uniform convergence on compact sets is obvious and we deduce the convergence in distribution in the Skorokhod space endowed with the J_1-Borel σ-field. Since convergence in distribution to a constant in a metric space implies convergence in probability, we deduce (2.21) and thus (2.15) in any cases.

Step 3. We first note that it suffices to prove for every $\ell \in \{\mathtt{u}, \mathtt{d}\}$,

$$\sup_{0\leqslant t\leqslant v} \frac{\tau^{\ell}_{N(ut)+1}}{a \circ s(u)} \xrightarrow[u\to\infty]{\mathbb{P}} 0. \tag{2.25}$$

Besides, it follows from **Step 2**. that given $\delta > 0$, there exists $s \geqslant 0$ such that for all $u \geqslant s$, B: beaucoup de 's'

$$\mathbb{P}\left(\Omega_{u,\delta} := \left\{\sup_{0\leqslant t\leqslant v}\left|\frac{N(ut)}{s(u)} - t\right| \leqslant \delta\right\}\right) \geqslant 1-\delta. \tag{2.26}$$

In addition, one can see that for every $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant v}\frac{\tau^{\ell}_{N(ut)+1}}{a\circ s(u)} > \varepsilon\ ,\ \Omega_{u,\delta}\right) \leqslant 1-\left[1-\mathbb{P}(\tau^{\ell} > \varepsilon\, a\circ s(u))\right]^{2\delta s(u)}. \tag{2.27}$$

Furthermore, it comes from Appendix A.1 that there exists a positive constant c such that

$$u\,\mathbb{P}(\tau^{\ell} > \varepsilon a(u)) \underset{u\to\infty}{\sim} c\left(\frac{2-\alpha}{\alpha}\right)\varepsilon^{-\alpha}. \tag{2.28}$$

Therefore, we deduce that for every $\delta > 0$,

$$\limsup_{u\to\infty}\mathbb{P}\left(\sup_{0\leqslant t\leqslant v}\frac{\tau^{\ell}_{N(ut)+1}}{a\circ s(u)} > \varepsilon\right) \leqslant \delta + \left[1-\exp\left(-2\delta k\left(\frac{2-\alpha}{\alpha}\right)\varepsilon^{-\alpha}\right)\right]. \tag{2.29}$$

Letting $\delta \longrightarrow 0$ ends the proof.

Step 4.To keep it concise, we only give details for the case of $\alpha = 1$ and $\mathbf{d}_{T} < \infty$, the other situations being simpler. Setting $s(u) = u/\mathbf{d}_{T}$, it follows from (2.14) that we can apply the continuous mapping theorem [10, Theorem 5.1, p.30] as in [10, Sect. 17., pp. 143-150] and deduce

$$\left\{\frac{\widetilde{C}_{N(ut)} - (\mathbf{d}_{S}-\mathbf{b}_{S})\,ut}{a\circ s(u)}\right\}_{n\geqslant 0} \overset{\mathrm{J}_1}{\underset{u\to\infty}{\Longrightarrow}} \{\mathfrak{C}(t)\}_{t\geqslant 0}. \tag{2.30}$$

Then, looking at the decomposition (2.12), we obtain the desired result by using the Slutsky's theorem together with **Step 3**.. This ends the proof, excepted for Lemmas 2.2 and 2.1. □

Proof of Lemma 2.2. First, we can show by using [18, pp. 170-174] that a suitable centering term (in the numerator) to get the marginal convergence at time $t = 1$ above is

$$b(u) := u\left(\int_0^{k(u)}\mathbb{P}(w > s)ds - \int_0^{k(u)}\mathbb{P}(-w > s)ds\right), \tag{2.31}$$

which can be rewritten when w is integrable as

$$b(u) := u\left[\mathbb{E}[w] - \left(\int_{k(u)}^{\infty}\mathbb{P}(w > s)ds - \int_{k(u)}^{\infty}\mathbb{P}(-w > s)ds\right)\right]. \tag{2.32}$$

In fact, the convergence holds when u runs through the integers in most of the papers but since $b(u)$ and $k(u)$ are regularly varying (of indices 0 and 1 respectively) a Slutsky type argument implies the convergence for u along the real numbers. Besides, it is well known that the functional convergence follows from the convergence of the marginal at $t = 1$.

To go further, we can see that for t sufficiently large there exist $\lambda^{\pm}(t)$ such that

$$\int_t^{\infty}\mathbb{P}(\pm w > s)ds = \frac{1}{2}\int_{\lambda^{\pm}(t)}^{\infty}\mathbb{P}(|w| > s)ds \quad \text{or} \quad \int_0^{t}\mathbb{P}(\pm w > s)ds = \frac{1}{2}\int_0^{\lambda^{\pm}(t)}\mathbb{P}(|w| > s)ds, \tag{2.33}$$

depending on whether w is integrable or not. Moreover, since the right tail and the left tail of w are well balanced, standard results on regularly functions (in particular those of slowly variations) and their inverses implies that $\lambda^{\pm}(t)/t$ tend to 1 as t goes to infinity. Furthermore, noting that the two-sided tail of w is regularly varying of index 1, the de Haan theory applies – especially [8, Theorem 3.7.3, pp. 162-163] coupled with [8, Theorem 3.1.16, p. 139] – and with (2.33) it implies that

$$\int_u^{\infty} \mathbb{P}(w>s)ds - \int_u^{\infty} \mathbb{P}(-w>s)ds \quad \text{or} \quad \int_0^u \mathbb{P}(w>s)ds - \int_0^u \mathbb{P}(-w>s)ds, \tag{2.34}$$

depending on whether w is integrable or not, is negligible with respect to $u\mathbb{P}(|w|>u)$ as u goes to infinity. To conclude, it suffices to note that $u\mathbb{P}(|w|>k(u))$ is equivalent to 1 in a neighbourhood of infinity and thus by the Slutsky's theorem we deduce convergences (2.18). □

Proof of Lemma 2.1. In the sequel, we refer to Appendix A.1 regarding the first two estimates in (2.1) as well as its right-hand side when $\mathbf{d}_{\tau} < \infty$. Hence, we focus on the situation when $\mathbf{d}_{\tau} = \infty$ (and thus $\alpha = 1$). Consider the tail distribution $\mathscr{T}(t)$ introduced in (A.4). It is not difficult to see that a suitable choice of the slowly varying functions can be achieved by setting $\Xi_1(u) := a(u)\mathscr{T} \circ a(u)$ and $d(u) := \Theta \circ a \circ s(u)$ respectively. Applying [8, Proposition 1.5.9a., p. 26] it comes that $u\mathscr{T}(u)$ is negligible with respect to $\Theta(u)$ as u goes to infinity and the right-hand side of (2.1) follows. □

3 The anomalous situations

Let us start this section by describing the behaviour of the normalizing function.

Lemma 3.1. *For every $\alpha \in (0,1)$, one has*

$$\lambda(u) \underset{u\to\infty}{\sim} \frac{(2-\alpha)(1-\alpha)}{\alpha} u. \tag{3.1}$$

We detail the functional convergence (1.16) when the limit process $\mathscr{S}_\alpha$ is no longer a stable Lévy process nor even a Markov process. Roughly speaking, the latter is a random continuous piecewise linear function built from an α-stable subordinator T_α (with no drift) as follows.

1) To each random excursion intervals $I = (T_\alpha(u-), T_\alpha(u))$ is attached a Rademacher random variable $\mathscr{X}_I$ of parameter $(1+\mathbf{b}_S)/2$, all of them being independent from each other and of T_α.

2) The slope of $\mathscr{S}_\alpha$ on each jump interval I is chosen as $\mathscr{X}_I$.

Such a stochastic process is properly exposed in [4] and its construction is briefly recalled in section A.2 of the Appendix. Define the so called label process $\{\mathscr{X}_\alpha(t)\}_{t\geqslant 0}$ by $\mathscr{X}_\alpha(t) := \mathscr{X}_I$ when $t \in I$, no matter its values elsewhere, and consider the associated age time process given by

$$\mathscr{A}_\alpha(t) := \sup\{v \geqslant 0 : \mathscr{X}_{t-v} = \mathscr{X}_t\}. \tag{3.2}$$

Proposition 3.1. *The distribution of $\{\mathscr{S}_\alpha(t)\}_{t\geqslant 0}$ does not depend on the scale parameter chosen for T_α and for all $\lambda > 0$,*

$$\{\mathscr{S}_\alpha(\lambda t)\}_{t\geqslant 0} \overset{\mathscr{L}}{=} \{\lambda \mathscr{S}_\alpha(t)\}_{t\geqslant 0}. \tag{3.3}$$

To go further, this stochastic process is continuous, of bounded variation and it can be written as

$$\mathscr{S}_\alpha(t) = \int_0^t \mathscr{X}_\alpha(s)\, ds. \tag{3.4}$$

Furthermore, the stochastic process $\{(\mathscr{X}_\alpha(t), \mathscr{A}_\alpha(t))\}_{t\geqslant 0}$ on $\{-1,1\} \times [0,\infty)$ is Markovian.

Remark 3.1. *At the sight of Remark 1.1, it turns out that the triplet ($\mathscr{S}_\alpha$, $\mathscr{X}_\alpha$, $\mathscr{A}_\alpha$) is somehow the continuous time counterpart of (S, X, A). In particular, the label process $\mathscr{X}_\alpha$ is in some sense a particular model for a continuous time VLMC.*

Theorem 3.1 (anomalous situation)**.** *For every $\alpha \in (0,1)$, the functional convergence (1.16) can be rewritten in a stronger form as*

$$\left\{ \frac{S_{ut}}{u} \right\}_{t\geqslant 0} \xrightarrow[u\to\infty]{\mathscr{C}} \{\mathscr{S}_\alpha(t)\}_{t\geqslant 0}. \tag{3.5}$$

Moreover, the marginal $\mathscr{S}_\alpha(t)$ has the density function $f_t(x)$ on $(-t,t)$ given by

$$f_t(x) = \frac{2\sin(\pi\alpha)}{\pi t} \frac{(t-x)^{\alpha-1}(t+x)^{\alpha-1}}{\mathbf{r}_s(t-x)^{2\alpha} + 2\cos(\pi\alpha)(t+x)^\alpha(t-x)^\alpha + \mathbf{r}_s^{-1}(t+x)^{2\alpha}}, \tag{3.6}$$

with $\mathbf{r}_s := (1+\mathbf{m}_s)/(1-\mathbf{m}_s)$. This is, up to an affine transformation, a so called arsine Lamperti distribution. Besides, the parameter $\mathbf{m}_s$ is nothing but its mean.

When $\alpha = 1/2$ and $\mathbf{m}_s = 0$, the distribution $f_t(x)$ is nothing but the push forward image by $x \longmapsto 2tx-1$ of the classical arcsine distribution, the law of the occupation time of the half-line, up to the time t, of a one-dimensional standard Brownian motion. In full generality, this distribution also appears, up to an affine transformation, as the limit of the mean sojourn time of a class of discrete time processes described by Lamperti in [5]. In addition, as explained in [20, 21] the latter is the law of the occupation time of the half-line for a skew Bessel process of dimension $2-2\alpha$ with skewness parameter $(1+\mathbf{b}_s)/2$ and from [22, Corollary 4.2., p. 343] we obtain that $f_1(x)$ is the density of

$$\mathbf{D}_\alpha := \frac{T_\alpha^{\mathrm{u}} - T_\alpha^{\mathrm{d}}}{T_\alpha^{\mathrm{u}} + T_\alpha^{\mathrm{d}}}, \tag{3.7}$$

where T_α^{u} and T_α^{d} are independent positive α-stable random variable whose symbols are respectively

$$\frac{1+\mathbf{b}_s}{2}\left(1 - i\tan\left(\frac{\pi\alpha}{2}\right)\right)|u|^\alpha \quad \text{and} \quad \frac{1-\mathbf{b}_s}{2}\left(1 - i\tan\left(\frac{\pi\alpha}{2}\right)\right)|u|^\alpha. \tag{3.8}$$

Therefore, the marginal convergence at time $t = 1$ in (3.5) can be interpreted as a kind of law of large number in distribution which extends the classical one in [1] – the expectations in the right-hand side of (1.8) being replaced by T_α^{u} and T_α^{d}. Finally, the terminology used for the so called mean drift $\mathbf{m}_s$ is justified by the convergence

$$\lim_{n\to\infty} \frac{S_n}{n} \overset{\mathbb{L}^1}{=} \mathbf{m}_s, \tag{3.9}$$

a direct consequence of the latter Theorem. Note also that $\mathbf{d}_\tau = \infty$ and $\mathbf{m}_s = \mathbf{b}_s$ in this section (we refer to Appendix A.1). The reason we used $\mathbf{m}_s$ rather than $\mathbf{b}_s$ or vice versa is motivated by the will to focus on different meanings – a mean drift or a balance term – according to the situations.

Again, the proof of Lemma 3.1 follows from the estimates in Appendix A.1 and its proof is omitted. We begin with Proposition 3.1 which lay the ground to Theorem 3.1. We recall that a rigorous construction of the limit process is given in Appendix A.2.

Proof of Proposition 3.1

First note that the distribution of $\mathscr{S}_\alpha$ does not depend on the scale parameter of T_α because of the scaling property of T_α. The same argument applies to show the self-similarity of $\mathscr{S}_\alpha$ given in (3.3).

Then, the integral representation (3.4) follows directly from the construction of the limit process. Indeed, let us denote by $\mathscr{I}_\alpha(t)$ the integral of the label process. We observe that $\mathscr{S}_\alpha(t)$ as $\mathscr{I}_\alpha(t)$ are linear on each excursion interval $I = (T_\alpha(u-), T_\alpha(u))$ with a slope given by the Rademacher label $\mathscr{X}_I$. Note also that for every $t \in \{T_\alpha(s) : s \geqslant 0\}$ one can write

$$\mathscr{S}_\alpha(t) = \mathscr{I}_\alpha(t) = Y_\alpha(t) = \sum_{u \leqslant t} \pm \Delta T_\alpha(u), \tag{3.10}$$

where the $\pm$ represent the random labels associated with the corresponding excursion intervals and Y_α is the overshoot limit defined in (A.16). Then, we easily deduce the equality for every $t \in [0, \infty)$.

It remains to show that $(\mathscr{X}_\alpha, \mathscr{A}_\alpha)$ is Markovian. Let $\mathbb{P}_0^\alpha$ be the distribution of such stochastic process. From the regenerative property of a stable subordinator, we get that $\{(\mathscr{X}_\alpha(t+s), \mathscr{A}_\alpha(t+s))\}_{t \geqslant 0}$ is distributed as $\mathbb{P}_0^\alpha$ for any $s \in \{T_\alpha(t) : t \geqslant 0\}$. Furthermore, let $\mathscr{H}_\alpha$ be the remaining life time introduced in (A.19). By using [23, Lemma 1.10, p. 15] one has

$$\mathbb{P}(\mathscr{H}_\alpha(t) \in dh \mid \mathscr{A}_\alpha(t) = a) = \frac{\alpha\, a^\alpha}{(a+h)^{\alpha+1}}\, dh, \tag{3.11}$$

for any $t > 0$. We denote this homogeneous kernel by $N(a; dh)$. Then, introduce for every $\ell \in \{-1, 1\}$ and $a \geqslant 0$ the distribution $\mathbb{H}^\alpha_{(\ell,a)}$ of the process on $\{-1,1\} \times [0,\infty)$ equal to $t \mapsto (\ell, a+t)$ up the random time H_a distributed as $N(a; dh)$. Therefore, by construction and the regenerative property, it turns out that the distribution of the stochastic process $\{(\mathscr{X}_\alpha(t+s), \mathscr{A}_\alpha(t+s))\}_{t \geqslant 0}$ is – almost surely – the push-forward image $G_*(\mathbb{H}^\alpha_{(\mathscr{X}_\alpha(s), \mathscr{A}_\alpha(s))} \otimes \mathbb{P}_0^\alpha)$ under the gluing map

$$G(w_1, w_2) := w_1(t) \mathbb{1}_{\{t < H_a\}} + w_2(t) \mathbb{1}_{\{t > H_a\}}. \tag{3.12}$$

The Markov property is then a simple consequence of this representation. $\square$

Remark 3.2. *Since the distribution of $(\mathscr{A}_\alpha(t), \mathscr{H}_\alpha(t))$ can be computed explicitly, $\mathscr{X}_\alpha$ and thus $\mathscr{S}_\alpha$ can be easily (numerically) generated. Moreover, it comes from [24, Theorem (3.2), p. 506] that the label process admit the invariant (infinite) measure given by*

$$\int_{\{\mathtt{u},\mathtt{d}\}} \int_0^\infty G_*(\mathbb{H}^\alpha_{(\ell,a)} \otimes \mathbb{P}_0^\alpha) \frac{1}{a^{\alpha+1}}\, da\, \mathscr{X}(d\ell), \tag{3.13}$$

where $\mathscr{X}(d\ell)$ is the distribution allocating the weights $(1+\mathbf{b}_s)/2$ and $(1-\mathbf{b}_s)/2$ to $\mathtt{u} = 1$ and $\mathtt{d} = -1$ respectively. That can be put into perspective with the result in [6] stating that a the double-infinite comb PRW admits an invariant probability measure if and only if the persistence times are both integrable

Proof of Theorem 3.1

First note that the tightness in (3.5) is obvious since the modulus of continuity of $\{S_{ut}/u\}_{t \geqslant 0}$ is equal to 1 almost surely. Hence, we only need to show the convergence of the finite-dimensional marginal distributions. To this end, it suffices to adapt the results in [4]. In that paper, the authors consider Lévy walks of the form

$$\mathfrak{S}(t) := \sum_{k=1}^{N(t)} \Lambda_k J_k + (t - T_{N(t)}) \Lambda_{N(t)+1}, \tag{3.14}$$

with

$$T_n := \sum_{k=1}^{n} J_k \quad \text{and} \quad N(t) := \max\{k \geqslant 1 : T_k \leqslant t\}. \tag{3.15}$$

The random moving times J_k and the jumps $\Lambda_k J_k$ are *i.i.d.*, the Λ_k being *i.i.d.* and independent of the jumping times and of unit size. For our model, one can interpret $\{S_n\}_{n \geqslant 0}$ as such Lévy walk for which the random moving times are given by $\tau^{\mathtt{d}}_{2n-1}$ or $\tau^{\mathtt{u}}_{2n}$ alternatively and with corresponding random jumps given by $-\tau^{\mathrm{d}}_{2n-1}$ or τ^{u}_{2n}. Hence, the directions are deterministic and equal to -1 or 1 alternatively. Even if their assumptions do not fit to this model, we can see that their results extend easily to our situation. Indeed, the central convergence [4, Theorem 3.4., pp. 4023-4024] becomes in our situation

$$\left(\frac{M_n}{a(n)}, \frac{T_n}{a(n)}\right) \xrightarrow[n\to\infty]{\mathscr{L}} (S_\alpha(1), T_\alpha(1)), \tag{3.16}$$

where T_α and S_α defined in (A.15) are the coupled α-stable Lévy processes introduced to define the arcsine Lamperti anomalous diffusion. Thereafter, it is not difficult to check that the same continuous mapping and topological arguments used in the proof of [4, Theorem 4.11., p. 4032] and its Corollary 4.14., p. 4033, hold to obtain the functional convergence (3.5).

Remark 3.3. *As a matter of facts, the limit process given in (1.16) and that in (3.5) satisfy*

$$Z_\alpha(t) = \frac{\mathscr{S}_\alpha(t) - \mathbf{m}_s t}{c_\alpha}, \tag{3.17}$$

where c_α is the multiplicative constant in (3.1).

Let us explicit the marginal distributions. First, it comes that $\mathscr{S}_\alpha$ can be achieved as the scaling limit of a "true" Lévy walk, in the sense of [4]. The latter has random moving times and jumps respectively given by $\xi_n\tau^{\mathrm{u}}_n + (1-\xi_n)\tau^{\mathrm{d}}_n$ and $\xi_n\tau^{\mathrm{u}}_n - (1-\xi_n)\tau^{\mathrm{d}}_n$, with *i.i.d.* random directions $2\xi_n - 1$ independent of the running times and distributed as $\mathscr{X}(d\ell)$ given in Remark 3.2.

Remark 3.4. *The latter true Lévy walk is a randomized version of the PRW which is somehow more convenient to study, as it is already appeared for the recurrence and transience criteria in [1].*

It is also shown in [4, Theorem 5.6., pp. 4036-4037] that the density distribution $f_t(x)$ satisfies, in a weak sense, the fractional partial differential equation

$$\left(\int_{\mathscr{A}} \left(\frac{\partial}{\partial t} + \ell\frac{\partial}{\partial t}\right)^\alpha \mathscr{X}(d\ell)\right) f_t(x) = \frac{1}{\Gamma(1-\alpha)}\frac{1}{t^\alpha}\int_{\mathscr{A}} \delta_{\ell t}(dx)\,\mathscr{X}(d\ell). \tag{3.18}$$

Besides, it follows from [4, Section 5., p. 4035] that $f_t(x)$ can be expressed as

$$f_t(x) = \frac{1}{\Gamma(1-\alpha)}\int_{\mathscr{A}}\int_0^t \frac{u_\alpha(x-\ell(t-s),s)}{(1-s)^\alpha}\,ds\,\mathscr{X}(d\ell), \tag{3.19}$$

where $u_\alpha(x,t)$ denotes the 0-potential (the mean occupation time) density of (S_α, T_α). Furthermore, the Fourier-Laplace Transform (FLT) of $u_\alpha(x,t)$ can be computed explicitly and it leads to that associated with $\mathscr{S}_\alpha(1)$ given in equation (5.5) of [4]. It is equal here to

$$\frac{(s-iy)^{\alpha-1} + (s+iy)^{\alpha-1}}{\left(\frac{1+\mathbf{b}_s}{2}\right)(s-iy)^\alpha + \left(\frac{1-\mathbf{b}_s}{2}\right)(s+iy)^\alpha}. \tag{3.20}$$

Remark 3.5. *By using the results in [15], it may be possible to provide the same representations of the finite-dimensional marginal distributions.*

Taking the derivative at the origin with respect to the spacial variable in (3.20), we can see that the mean distribution of $f_1(x)$ is necessarily equal to $\mathbf{m}_s$. Furthermore, it seems conceivable to invert directly this FLT to get the expression (3.6). However, this is not the path we choose to borrow. In place, we employ and generalize the ideas of [5] where the Lamperti distributions appear as mean occupation time densities of a large class of stochastic processes. Finally, we make the connection with the theory of excursions.

Lamperti investigates in [5] the general question of the distribution of the mean occupation time of a set A for some stochastic process $\{\mathbf{X}_n\}_{n\geqslant 0}$ on a state space E. Regarding the dynamics, it is surprisingly only assumed that E can be divided into two sets – the aforementioned A and an other one B – plus a recurrent state σ in such a way that, if $\mathbf{X}_{n-1} \in A$ and $\mathbf{X}_{n+1} \in B$ or vice versa, then necessarily $\mathbf{X}_n = \sigma$. Given such process starting from σ, Lamperti consider N_n the occupation time of A up to the time n – the state σ being counted when the process comes from A – and he introduces $F(x)$ the generating function associated with the recurrence time of σ. This decomposition of the state space and how to count the occupation time is resumed by the weighted diagram in the left-hand side of Figure 3.2. The following Theorem is stated and proved in its paper.

Theorem 3.2. *The mean occupation time $\{N_n/n\}_{n\geqslant 1}$ converges in distribution as n goes to infinity to a non-degenerate limit if and only if there exist α and p in $(0,1)$ such that*

$$\lim_{n\to\infty} \mathbb{E}[N_n/n] = p \quad and \quad \lim_{x\uparrow 1}(1-x)F'(x)/(1-F(x)) = \alpha. \tag{3.21}$$

Besides –in that case – the limit distribution has for density on $(0,1)$ the function

$$\frac{\sin(\pi\alpha)}{\pi} \frac{t^{\alpha-1}(1-t)^{\alpha-1}}{rt^{2\alpha} + 2\cos(\pi\alpha)t^{\alpha}(1-t)^{\alpha} + r^{-1}(1-t)^{2\alpha}}, \tag{3.22}$$

with $r := (1-p)/p$.

The proof is based on a precise asymptotic estimate of the moments associated with N_n as n goes to infinity – via suitable generating functions and their regular variations – in order to identify the limit of the Stieljes transform of N_n/n.

In our case, it would seem natural to see A as the rises of S so that it can be write more or less as $S_n = 2N_n - n$. Unfortunately, we have not been able to fit this situation to the latter Lamperti dynamic. As a matter of fact, the communication diagram and the way to compute the suitable occupation time N_n can be given as in the right hand side of Figure 3.2. To be more precise, if we consider the 2-letters process $\mathbf{X}_n := X_nX_{n+1}$ with the decomposition of $\{\mathtt{u},\mathtt{d}\}\times\{\mathtt{u},\mathtt{d}\}$ given by $A := \{\mathtt{uu}\}$, $B := \{\mathtt{dd}\}$, $\sigma := \mathtt{ud}$ and $\eta := \mathtt{du}$ and if we denote by N_n the occupation time of A up to the time n – the state η being counted or not according to well chosen weights – the PRW satisfies for any $n \geqslant 1$ the relation $S_n = 2N_{n-1} - n$.

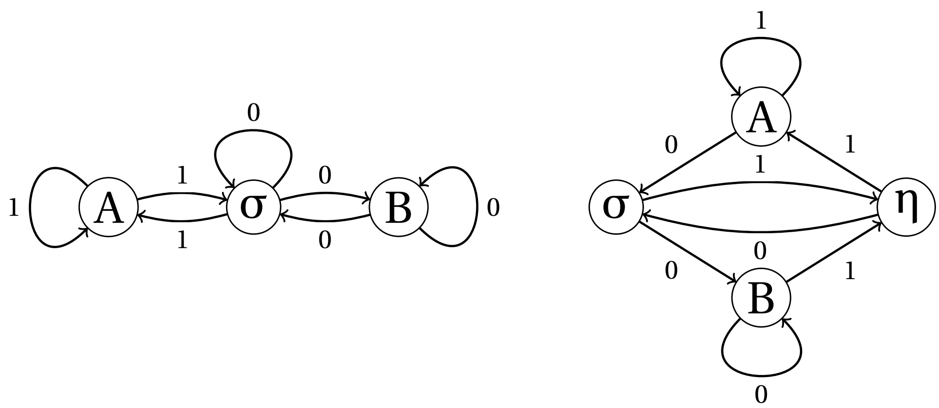

Figure 3.2: Original and current Lamperti decomposition of the state space

Note also that σ and η are recurrent states and since the random times spend in A and B are distributed as $\tau^{\mathtt{u}}$ and $\tau^{\mathtt{d}}$ respectively the recurrence times of σ and η are both equal in law to $\tau = \tau^{\mathtt{u}} + \tau^{\mathtt{d}}$.

To get the limit distribution, we adapt and follow a part of the proof in [5]. To this end, we introduce $p_{n,k}$ the probability – starting from σ – that $N_n = k$ and we denote by u_n and d_n the probability that $\tau^{\mathtt{u}} = n$ and $\tau^{\mathtt{d}} = n$ respectively. Thereafter, by conditioning with respect to the complete three events $\{\tau \leqslant n\} \sqcup \{\tau \geqslant n+1, \tau^{\mathtt{d}} \leqslant n\} \sqcup \{\tau^{\mathtt{d}} \geqslant n+1\}$ we obtain

$$p_{n,k} = \sum_{\substack{m+l\leqslant n\\ m\leqslant k}} d_l\, u_m\, p_{n-(m+l),k-m} + \sum_{l\leqslant n} d_l\, \mathscr{T}_{\mathtt{u}}(n-l)\delta_{k,n-l+1} + \mathscr{T}_{\mathtt{d}}(n)\delta_{k,0}, \tag{3.23}$$

with $\delta_{i,j}$ is 1 or 0 according to $i = j$ or not. Then, let $F_\ell(z)$ be the generating function of τ^ℓ and $T_\ell(z)$ be the one associated with its tail distribution. it follows that the double generating function associated with the latter difference equation satisfies

$$P(x,y) := \sum_{n,k\geqslant 0} p_{n,k}\, x^n y^k = \frac{F_{\mathtt{d}}(y)T_{\mathtt{u}}(xy)y + T_{\mathtt{d}}(x)}{1 - F_{\mathtt{d}}(x)F_{\mathtt{u}}(xy)}, \tag{3.24}$$

From our stability assumption and classical results on regular variations, we get that there exists a slowly varying function $L(z)$ such that

$$1 - F_{\mathtt{u}}(z) = (1-z)T_{\mathtt{u}}(z) \underset{z\uparrow 1}{\sim} \left(\frac{1+\mathbf{b}_s}{2}\right) L\left(\frac{1}{1-z}\right)(1-z)^\alpha, \tag{3.25}$$

the same asymptotic and equality being also true replacing $\mathtt{u}$ by $\mathtt{d}$ and $(1+\mathbf{b}_s)/2$ by $(1-\mathbf{b}_s)/2$. Then we can check that for any $\lambda > 0$,

$$\lim_{x\uparrow 1}(1-x)P\left(x, e^{-\lambda(1-x)}\right) = \frac{(1+\lambda)^{\alpha-1} + \mathbf{r}_s}{(1+\lambda)^\alpha + \mathbf{r}_s}, \tag{3.26}$$

and thereafter the proof follows exactly the same lines as [5]. We obtain the arcsine Lamperti density. Note that it may be possible to state and prove a general theorem for such Lamperti processes.

□

Remark 3.6. *One can deduce from (3.4) that [22, section 4.] and [20, Theorem 1] apply to our situation to get the latter marginal densities, making the link with the distribution of the occupation time of the positive half-line for a skew Bessel process.*

4 Some generalizations

In this section, we discuss briefly of some generalisation of the main theorem. First, following the ideas of [1, Section 4.], we extend the results to a wider class of PRWs in the case of persitence times with infinite mean. For these PRWs, there is no longer a renewal scheme allowing the cutting into independent pieces. Secondly, we say some words on the assumption 1.1 and more precisely on the extremal case.

Consider a double-infinite comb and attach to each finite leaf c another context tree $\mathbb{T}_c$ (possibly trivial) as in Figure 4.3. The leaves of the related graft are denoted by $\mathscr{C}_c$ and this one is endowed with Bernoulli distributions $\{q_l : l \in \mathscr{C}_c\}$ on $\{\mathtt{u},\mathtt{d}\}$. Note that any probabilized context tree on $\{\mathtt{u},\mathtt{d}\}$ can be constructed in this way. We denote by S^{g} the corresponding PRW. In this case, the random walk is particularly persistent in the sense that the rises and descents are no longer independent. A renewal property may still hold but it is more tedious to expect in general. Let $\underline{S}$ and $\overline{S}$ be the double-infinite comb PRWs with respective transitions

$$\underline{\alpha}_n^{\mathtt{u}} := \sup\{q_c(\mathtt{d}) : c \in \mathscr{C}_{\mathtt{u}^n\mathtt{d}}\}, \quad \underline{\alpha}_n^{\mathtt{d}} = \inf\{q_c(\mathtt{u}) : c \in \mathscr{C}_{\mathtt{d}^n\mathtt{u}}\},$$
$$\text{and} \quad \overline{\alpha}_n^{\mathtt{u}} := \inf\{q_c(\mathtt{d}) : c \in \mathscr{C}_{\mathtt{u}^n\mathtt{d}}\}, \quad \overline{\alpha}_n^{\mathtt{d}} := \sup\{q_c(\mathtt{u}) : c \in \mathscr{C}_{\mathtt{d}^n\mathtt{u}}\}, \tag{4.1}$$

and denote for every $\ell \in \{\mathtt{u}, \mathtt{d}\}$ by $\underline{\tau}^{\ell}$ and $\overline{\tau}^{\ell}$ the corresponding waiting times.

Corollary 4.1. *Assume that $\underline{S}$ or $\overline{S}$ satisfy Assumption 1.1 with $\alpha \in (0,1)$ but also that $\underline{\tau}^{\ell}$ and $\overline{\tau}^{\ell}$ have asymptotically equivalent tail distributions for each $\ell \in \{\mathtt{u}, \mathtt{d}\}$. Then Theorem 3.1 holds for S^{g}.*

The proof of this Corollary is a straightforward consequence of the comparison results stated in [1]. For instance, those assumptions can be easily achieved when the non-trivial grafts are both in finite number and of finite size.

Finally, for the seek of simplicity, we have exclude in Assumption 1.1 the extremal mean drifts, say $\mathbf{m}_S = 1$ for the example. This situation may arise when $\alpha = 1$ and $\mathbf{d}_T = \infty$ but also when $\alpha \in (0,1)$. As a matter of fact, it is possible to obtain similar functional convergences as (2.4) or (3.5) but toward the null or the identity process respectively. To this end, it suffices to replace $\Sigma(t)^2$ by $V(t)$ defined in (A.4) in the settings of $a(u)$ and the proofs follows the same lines. The additional assumptions to obtain non trivial limits are more subtle. If $\tau^{\mathtt{d}}$ belongs to $\mathrm{D}(\gamma)$ with $0 < \gamma \leqslant \alpha < 1$ – assumption $(\#)$ – or if it is a relatively stable distribution (see [8, Chap. 8.8, p. 372]) – assumption $(\star)$ – then the generic form of the functional convergence seems to be respectively

$$\left\{ -\frac{u^{1-\frac{\alpha}{\gamma}}}{\Xi(u)} \left(\frac{S_{\lfloor ut \rfloor}}{u} - t \right) \right\}_{t \geqslant 0} \xrightarrow[u \to \infty]{\mathrm{J}_1} \left\{ T^{\mathtt{d}}_{\gamma} \circ N^{\mathtt{u}}_{\alpha}(t) \right\}_{t \geqslant 0} \quad (\#),$$

$$\text{or} \quad \left\{ -\frac{u^{1-\alpha}}{\Xi(u)} \left(\frac{S_{ut}}{u} - t \right) \right\}_{t \geqslant 0} \xrightarrow[u \to \infty]{\mathscr{C}} \left\{ N^{\mathtt{u}}_{\alpha}(t) \right\}_{t \geqslant 0} \quad (\star), \tag{4.2}$$

where $\Xi(u)$ is slowly varying and $T^{\mathtt{d}}_{\gamma}$ is a γ-stable subordinator independent – contrary to the non-extremal situation – of the α-local time $N^{\mathtt{u}}_{\alpha}$. Besides, assuming in place of assumption $(\star)$ that $\tau^{\mathtt{d}} \in \mathrm{D}(\gamma)$ for some $\gamma \in [1,2]$, the next term in the asymptotic expansion $(\star)$ in (4.2) is – heuristically – of the order of $S^{\mathtt{d}}_{\gamma} \circ N^{\mathtt{u}}_{\alpha}(t)/u^{\alpha/\gamma}$, where $S^{\mathtt{d}}_{\gamma}$ is a γ-stable Lévy process independent of $N^{\mathtt{u}}_{\alpha}$.

A Appendices

The estimates in Appendix A.1 are standards and rely on classical results on stable distributions and regularly varying functions (see [8, 9] for instance). Regarding the construction of the anomalous diffusion in Appendix A.2, we mention [4] for the original definition and [23] concerning classical results about Lévy subordinators.

A.1 Equivalent settings of Assumption 1.1 and some consequences

Let $\mathscr{T}_{\mathtt{c}}(t)$ and $V_{\mathtt{c}}(t)$ be the two-sided distribution tail and the truncated second moment of $\tau^{\mathtt{c}}$. In the following, we denote by $\mathrm{RV}(\lambda)$ the set of regularly varying functions of index $\lambda \in \mathbb{R}$, also named the set

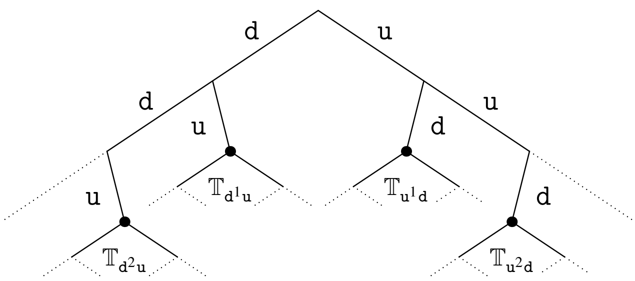

Figure 4.3: Grafting of the double-infinite comb

of slowly varying functions SV if $\lambda = 0$. It is well known that hypothesis (1.12) is equivalent to

$$V_{\mathtt{c}}(t) \in \mathrm{RV}(2-\alpha), \quad \text{or equivalently when } \alpha \neq 2, \quad \mathscr{T}_{\mathtt{c}}(t) \in \mathrm{RV}(-\alpha), \tag{A.1}$$

with an additional tail balance criterion when $\alpha \neq 2$: there exists $\beta \in [-1,1]$ with

$$\lim_{t\to\infty} \frac{\mathbb{P}(\tau^{\mathtt{c}} > t) - \mathbb{P}(\tau^{\mathtt{c}} < -t)}{\mathbb{P}(\tau^{\mathtt{c}} > t) + \mathbb{P}(\tau^{\mathtt{c}} < -t)} = \beta. \tag{A.2}$$

Also, it is well known – see [9, Theorem 2, Chap. VIII.9, p. 283] for instance – that

$$\lim_{t\to\infty} \frac{t^2 \mathscr{T}_{\mathtt{c}}(t)}{V_{\mathtt{c}}(t)} = \frac{2-\alpha}{\alpha}, \tag{A.3}$$

Finally, we introduce

$$\mathscr{T}(t) := \mathscr{T}_{\mathtt{u}}(t) + \mathscr{T}_{\mathtt{d}}(t) \quad \text{and} \quad V(t) := V_{\mathtt{u}}(t) + V_{\mathtt{d}}(t), \tag{A.4}$$

and given any functions $f(t)$ and $g(t)$ defined on a neighbourhood of infinity, we set $f(t) \asymp g(t)$ when there exists $c > 0$ such that for t sufficiently large $c^{-1} g(t) \leqslant f(t) \leqslant c g(t)$.

The proofs of the following two Lemma are straightforward calculations involving only classical results on regularly varying functions or standard estimations, they are omitted. The first one can be useful, among other considerations, to state alternative forms of the stability condition 1.12, whereas the second one allow us to discriminate whether the persistence times are integrable or not and thus to identify the mean drift in Assumption 1.1.

Lemma A.1. *Assuming the existence of the mean drift $\mathbf{m}_s \in (-1,1)$, the stability hypothesis (1.12) is then equivalent when $\alpha \neq 2$ to $\tau^{\mathtt{u}} - \tau^{\mathtt{d}} \in \mathrm{D}(\alpha)$ but also to*

$$\Big[V(t) \in \mathrm{RV}(2-\alpha) \quad \textit{or} \quad \mathscr{T}(t) \in \mathrm{RV}(-\alpha) \Big] \quad \textit{and} \quad \lim_{t\to\infty} \frac{\mathscr{T}_{\mathtt{u}}(t) - \mathscr{T}_{\mathtt{d}}(t)}{\mathscr{T}_{\mathtt{u}}(t) + \mathscr{T}_{\mathtt{d}}(t)} = \mathbf{b}_s. \tag{A.5}$$

In that case, the functions $\mathscr{T}(t)$ and $V(t)$ are respectively the two-sided tail distribution and the truncated second moment of $\tau^{\mathtt{u}} - \tau^{\mathtt{d}}$ but also of $\tau^{\mathtt{u}} + \tau^{\mathtt{d}}$. Besides, one has

$$\mathscr{T}_{\mathtt{c}}(t) \underset{t\to\infty}{\sim} \left[(1-\mathbf{m}_s)^{\alpha} \left(\frac{1+\mathbf{b}_s}{2} \right) + (1+\mathbf{m}_s)^{\alpha} \left(\frac{1-\mathbf{b}_s}{2} \right) \right] \mathscr{T}(t), \tag{A.6}$$

and a similar asymptotic between $V_{\mathtt{c}}(t)$ and $V(t)$. Moreover, the balance term β in (A.2) satisfies

$$\beta = \frac{(1-\mathbf{m}_s)^{\alpha}(1+\mathbf{b}_s) - (1+\mathbf{m}_s)^{\alpha}(1-\mathbf{b}_s)}{(1-\mathbf{m}_s)^{\alpha}(1+\mathbf{b}_s) + (1+\mathbf{m}_s)^{\alpha}(1-\mathbf{b}_s)}. \tag{A.7}$$

Finally, one has

$$\mathbb{V}[\tau_1^{\mathtt{c}} \mathbb{1}_{\{|\tau_1^{\mathtt{c}}| \leqslant t\}}] \underset{t\to\infty}{\sim} V_{\mathtt{c}}(t) \underset{t\to\infty}{\sim} \Sigma(t)^2. \tag{A.8}$$

One can draw the following consequences. First, when $\alpha \neq 2$, the stability condition (1.12) means that at least $\tau^{\mathtt{u}}$ or $\tau^{\mathtt{d}}$ belongs $\mathrm{D}(\alpha)$, the tail distribution of the other waiting time being comparable or negligible with the former. On the other hand, when $\alpha = 2$, we deduce from [25, Theorem 4.5., p. 790] the following consequence.

Remark A.1. *It is possible for every linear combination of $\tau^{\mathtt{u}}$ and $\tau^{\mathtt{d}}$, excepted $\tau^{\mathtt{c}}$, to not belong to the domain of attraction of a normal distribution.*

Finally, the functions $a(u)$, $s(u)$, along with $\Sigma(t)^2$ and $\Theta(t)$ and any others built from them by composition, product or generalized inverse, are regularly varying.

Lemma A.2. *Let μ be a positive measure on $[0,\infty)$ and for any $p \geqslant 0$ and $t \geqslant 0$,*

$$M_p(t) := \int_{[0,t]} x^p \mu(dx) \quad \textit{and} \quad T_p(t) := \int_{(t,\infty)} x^p \mu(dx). \tag{A.9}$$

Then for any $q > p \geqslant 0$,

$$M_p(t) \asymp \int_{[1,t]} \frac{M_q(u)}{u^{q-p+1}} du + \frac{M_q(t)}{t^{q-p}} \quad \textit{and} \quad T_p(t) \asymp \int_{(t,\infty)} \frac{M_q(u)}{u^{q-p+1}} du + \frac{M_q(t)}{t^{q-p}}. \tag{A.10}$$

Therefore, when $\alpha \in (1,2]$, the persistence times are both of finite mean (thus $\mathbf{d}_T < \infty$) and $\mathbf{m}_S = \mathbf{d}_S$ if we suppose Assumption 1.1. On the contrary, when $\alpha \in (0,1)$, the persistence times are both of infinite mean (thus $\mathbf{d}_T = \infty$) and $\mathbf{m}_S = \mathbf{b}_S$. To conclude, when $\alpha = 1$, the two latter situations are possible.

A.2 Construction of the Lamperti anomalous diffusion

Let T_α be an α-stable Lévy subordinator with no drift and let $\mathscr{J} \subset (0,\infty)$ be its random set of jumps. The closure of the image $\mathscr{R}_\alpha := \{T_\alpha(t) : t \geqslant 0\}$ is a perfect set of zero Lebesgue measure satisfying

$$\overline{\mathscr{R}_\alpha} = \mathscr{R}_\alpha \sqcup \{T_\alpha(u-) : u \in \mathscr{J}\} \quad \text{and} \quad [0,\infty) \backslash \overline{\mathscr{R}_\alpha} = \bigsqcup_{u \in \mathscr{J}} (T_\alpha(u-), T_\alpha(u)). \tag{A.11}$$

The backward recurrence time (also the current life time) and the forward renewal time (also the residual life time) are the stochastic processes respectively defined as

$$G_\alpha(t) := \sup\{s \leqslant t : s \in \mathscr{R}_\alpha\} \quad \text{and} \quad H_\alpha(t) = \inf\{s \geqslant t : s \in \mathscr{R}_\alpha\}. \tag{A.12}$$

We refer carefully to Figure A.4 below. Those are *càdlàg* and $G_\alpha(t) = T_\alpha(u-)$ and $H_\alpha(t) = T_\alpha(u)$ when $T_\alpha(u-) \leqslant t < T_\alpha(u)$ and $G_\alpha(t) = H_\alpha(t) = T_\alpha(u)$ when $t = T_\alpha(u)$. To go further, we consider the local time of the stable subordinator T_α defined by

$$N_\alpha(t) := \inf\{s > 0 : T_\alpha(s) > t\} = \sup\{s > 0 : T_\alpha(s) \leqslant t\}. \tag{A.13}$$

Note the useful switching identity $\{T_\alpha(s) \leqslant t\} = \{s \leqslant N_\alpha(t)\}$. It turns out that N_α is a non-decreasing continuous stochastic process such that the support of the random Stieljes measure dN_α is equal to $\overline{\mathscr{R}_\alpha}$ and $N_\alpha(t) = u$ whenever $T_\alpha(u-) \leqslant t \leqslant T_\alpha(u)$. When $\alpha = 1/2$, it is nothing but the classical local time of a Brownian motion. Furthermore, one can be decomposed T_α as

$$T_\alpha(t) = T_\alpha^{\mathrm{u}}(t) + T_\alpha^{\mathrm{d}}(t) := \left(\frac{1+\mathbf{b}_S}{2}\right)^{1/\alpha} T'_\alpha(t) + \left(\frac{1-\mathbf{b}_S}{2}\right)^{1/\alpha} T''_\alpha(t), \tag{A.14}$$

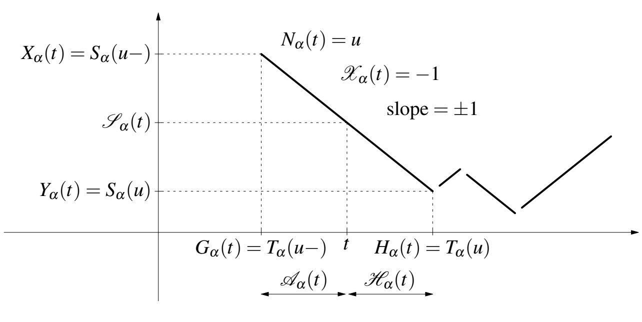

Figure A.4: Construction of the anomalous diffusion where T'_α and T''_α are *i.i.d.* with the same distribution as T_α. This can be obtained intrinsically by labelling each jump interval (excursion) $I = (T_\alpha(u-), T_\alpha(u))$ as in [22, pp. 342-343] by *i.i.d.* Rademacher random variables $\mathscr{X}_I$ independent of T_α and of parameter $(1+\mathbf{b}_s)/2$. It follows that $T^{\mathrm{u}}_\alpha(t)$ and $T^{\mathrm{d}}_\alpha(t)$ can be viewed as the sums up to the time t of the jumps $\Delta T_\alpha(u) := T_\alpha(u) - T_\alpha(u-)$ for which the corresponding labels are respectively equal to one and minus one. They are both thinning of the initial subordinator. Coupled with the latter decomposition, we can consider the α-stable Lévy process

$$S_\alpha(t) := T^{\mathrm{u}}_\alpha(t) - T^{\mathrm{d}}_\alpha(t) = \left(\frac{1+\mathbf{b}_s}{2}\right)^{1/\alpha} T'_\alpha(t) - \left(\frac{1-\mathbf{b}_s}{2}\right)^{1/\alpha} T''_\alpha(t). \tag{A.15}$$

It can be viewed as a pure jump process whose increment $\Delta S_\alpha(u)$ is equal to $\Delta T_\alpha(u)$ or $-\Delta T_\alpha(u)$ according to the label of the corresponding excursion. Following the terminology used in [16], we introduce the so-called lagging and leading *càdlàg* stochastic processes respectively defined by

$$X_\alpha(t) := [S^-_\alpha \circ N_\alpha]^+(t) \quad \text{and} \quad Y_\alpha(t) := S_\alpha \circ N_\alpha(t), \tag{A.16}$$

where $F^\pm(t) := F(t\pm)$ is the right or the left continuous version of a function $F(t)$. It is not difficult to check that $X_\alpha(t) = S_\alpha(u-)$ and $Y_\alpha(t) = S_\alpha(u)$ when $T_\alpha(u-) \leqslant t < T_\alpha(u)$ whereas $X_\alpha(t) = Y_\alpha(t) = S_\alpha(u)$ when $t = T_\alpha(u)$. Note also that the first and last renewal time processes can be rewritten as

$$G_\alpha(t) = [T^-_\alpha \circ N_\alpha]^+(t) \quad \text{and} \quad H_\alpha(t) = T_\alpha \circ N_\alpha(t). \tag{A.17}$$

Thereafter, we called arcsine Lamperti anomalous diffusion a stochastic process distributed as

$$\mathscr{S}_\alpha(t) := \begin{cases} X_\alpha(t) + \dfrac{t - G_\alpha(t)}{H_\alpha(t) - G_\alpha(t)}(Y_\alpha(t) - X_\alpha(t)), & \text{when} \quad t \notin \mathscr{R}_\alpha, \\ X_\alpha(t), & \text{when} \quad t \in \mathscr{R}_\alpha. \end{cases} \tag{A.18}$$

In other words, the so-called anomalous diffusion $\mathscr{S}_\alpha(t)$ is the center of mass of the lagging and leading processes $X_\alpha(t)$ and $Y_\alpha(t)$ with respective weights given by the so called remaining time and age time stochastic processes

$$\mathscr{H}_\alpha(t) := H_\alpha(t) - t \quad \text{and} \quad \mathscr{A}_\alpha(t) = t - G_\alpha(t). \tag{A.19}$$

Remark A.2. *It seems that [2, Theorem 1.3.] and equation (3.24), p. 3281, in the proof of [3, Theorem 2.6., p. 3272] are not entirely true. Indeed, their results would imply that*

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{\mathscr{L}} Y_\alpha(1). \tag{A.20}$$

However, the latter convergence can not be true since $Y_\alpha(1)$ is not compactly supported contrary to $X_\alpha(1)$. Moreover, in their results it misses the residual term insuring the continuity of the limit process. This kind of subtle mistake has already been made in [13] but corrected in [14].

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